Lasing and superradiant phases of the driven dissipative Dicke model.

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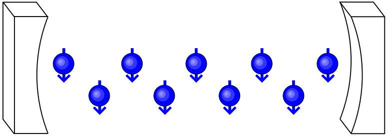
FOUNDED 1413

SUPA

PCS, October 2017

Dicke model and Dicke-Hepp-Lieb transition



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- Coherent state: $|\Psi\rangle \rightarrow e^{\lambda \hat{a}^\dagger + \eta S^+}|\Omega\rangle$
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[Hepp, Lieb, Ann. Phys. '73]

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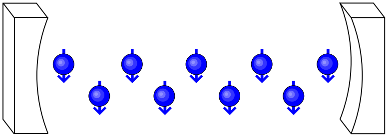

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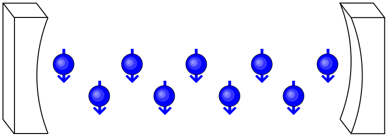

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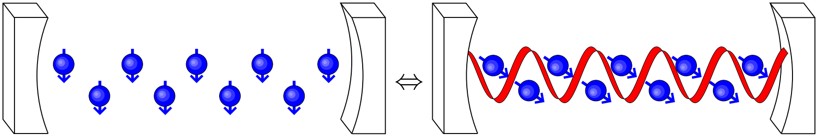

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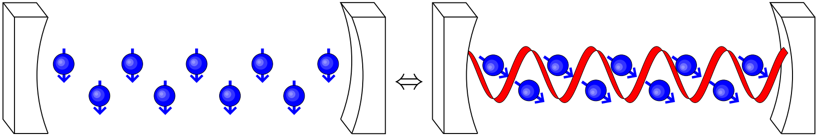

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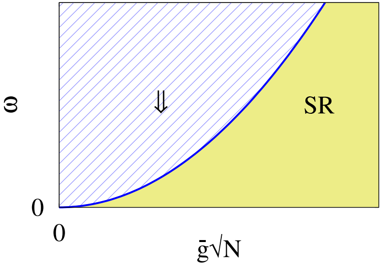

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“Textbook” Laser: Maxwell Bloch equations

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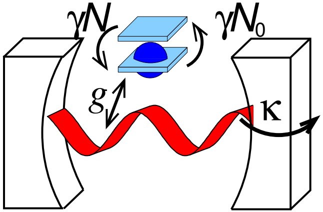

- $|\alpha|^2 > 0$ if $2N_0 g^2 > \gamma\kappa$
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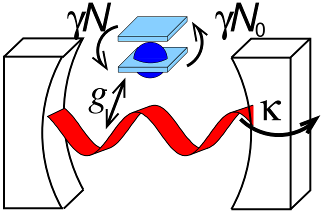

Maxwell-Bloch eqns: $\alpha = \langle \hat{a} \rangle$, $P = -i\langle \sigma^- \rangle$, $N = 2\langle \sigma^z \rangle$

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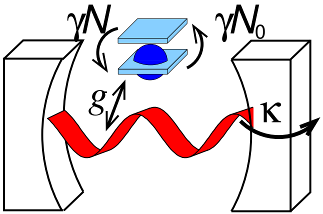

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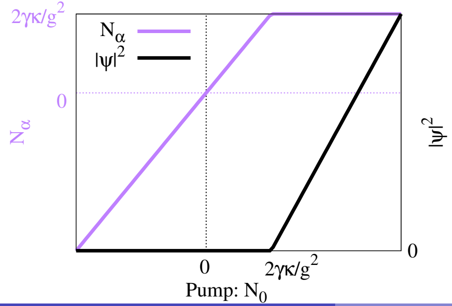

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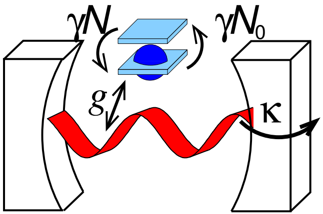

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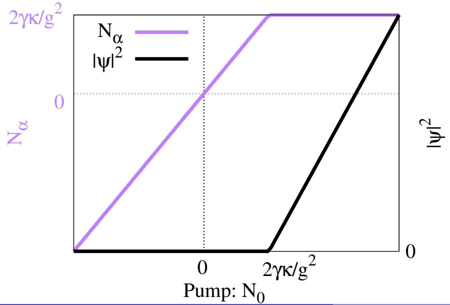

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“Textbook” Laser: Semiclassical equations

- Semiclassical laser theory $n = \langle \hat{a}^\dagger \hat{a} \rangle$

$$\partial_t n = \gamma N_0 \frac{2g^2(n+1)}{\gamma\gamma_t + 4g^2(n+1)} - \kappa n$$

  - MF Transition at $N_0 2g^2/\gamma_t = \kappa$
  - No symmetry breaking
- Spontaneous emission: finite “size” corrections

$$n = \frac{1}{2\beta}\left[\frac{N_0}{N_c} - 1 \pm \sqrt{\left(\frac{N_0}{N_c} - 1\right)^2 + 4\beta\frac{N_0}{N_c}}\right]$$

$\beta \simeq 4g^2/\gamma\gamma_t$

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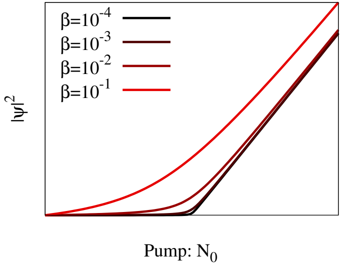

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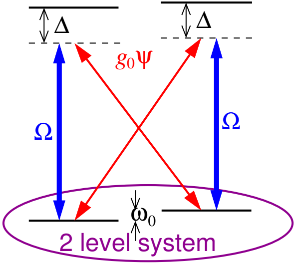
Raman driven Dicke model [Dimer *et al.* PRA '07]



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- Coupling $g = \frac{g_0 \Omega}{2\Delta}$
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- Imbalanced case (internal states):

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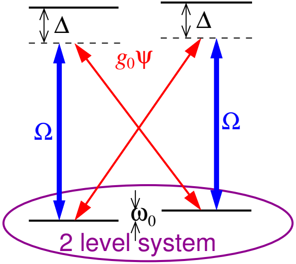

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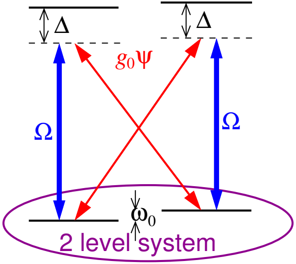

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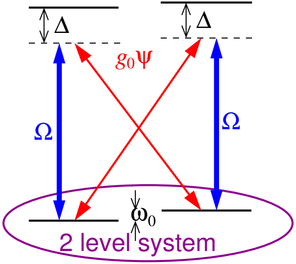

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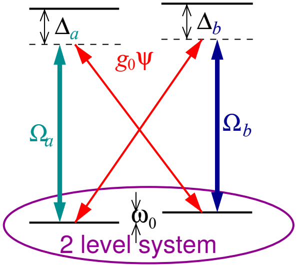

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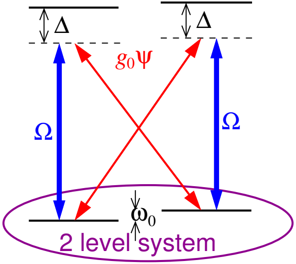

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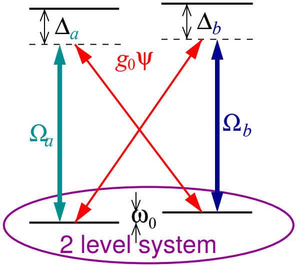
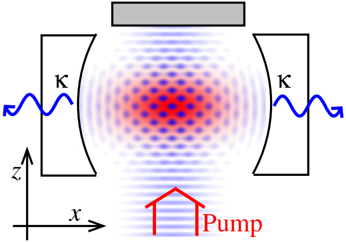


Open Dicke model theory

- Momentum degrees of freedom:
  $\psi = \psi_{\Downarrow} + \psi_{\Uparrow} \cos(kx)\cos(kz)$
- Effective 2LS $(\psi_{\Downarrow}, \psi_{\Uparrow})$

$$H_{\text{eff}} = \omega \hat{a}^{\dagger}\hat{a} + \sum_n \frac{\omega_0}{2}\sigma_n^z + g_{\text{eff}}\sigma_n^x(\hat{a} + \hat{a}^{\dagger}) + U\sigma_n^z\hat{a}^{\dagger}\hat{a}$$



- Extra "feedback" term U, cavity loss κ
- Single mode – mean-field EOM, $\alpha = \langle \hat{a} \rangle$, $S = \sum_n \sigma_n/2$.

$$\dot{S}^- = -i(\omega_0 + U|\alpha|^2)S^- + 2ig_{\text{eff}}(\alpha + \alpha^*)S^z$$
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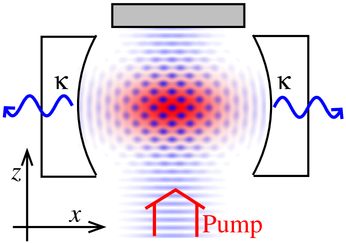

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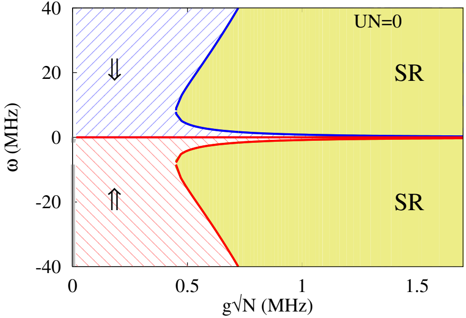
Classical dynamics

Changing U:

$U = 0$

$U < 0$

$U > 0$



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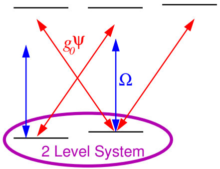
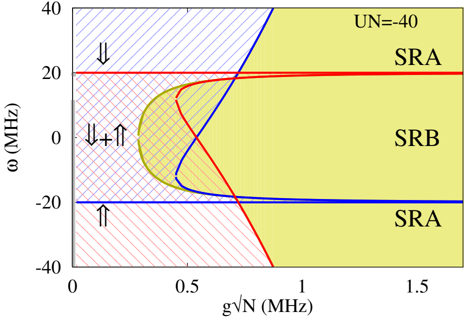


$$U \propto \frac{g_0^2}{\omega_c - \omega_a}$$



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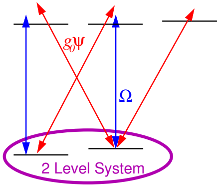

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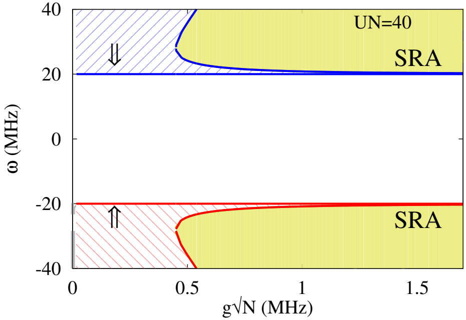


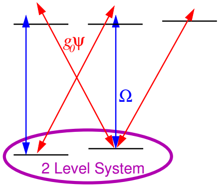
Classical dynamics

Changing U:

$U = 0$

$U < 0$

$U > 0$



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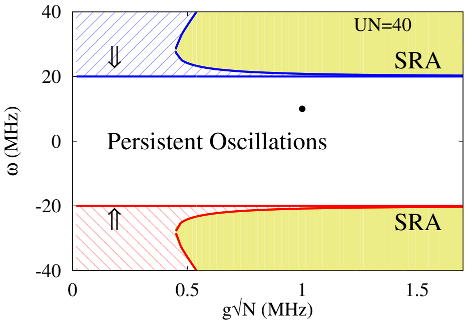
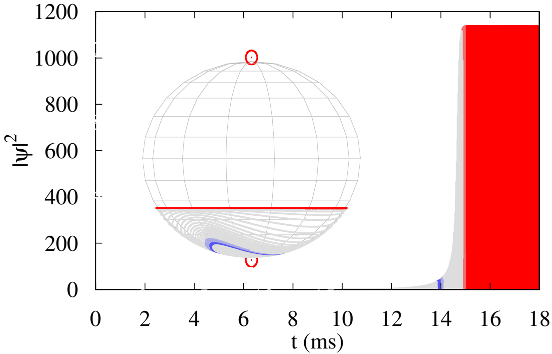


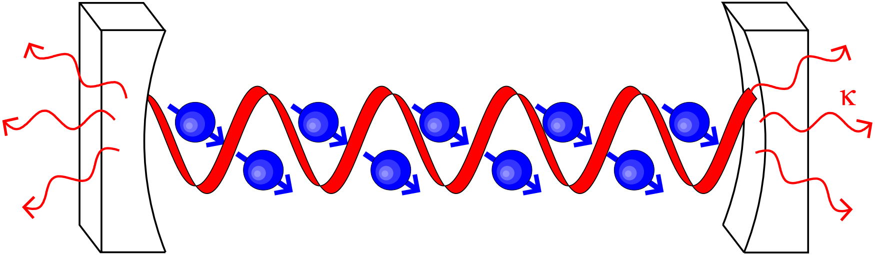


1. Introduction: Open Dicke model reminder
   - Dicke superradiance vs lasing
   - Driven Dicke model

2. Behaviour with dephasing
   - Mean field theory problem
   - Exact solution and cumulant expansion

3. Lasing vs superradiance: competition
   - Basic phase diagram
   - Signatures of states
   - Blue detuned pump

Individual dephasing



- Extra loss terms

$$\partial_t \rho = -i[H, \rho] + \kappa \mathcal{L}[\hat{a}] + \sum_\alpha \Gamma_\downarrow \mathcal{L}[\sigma_\alpha^-] + \Gamma_\phi \mathcal{L}[\sigma_\alpha^z]$$

$$\mathcal{L}[X] = X\rho X^\dagger - (X^\dagger X \rho + \rho X^\dagger X)/2$$

- $\Gamma_\downarrow, \Gamma_\phi$ break **S** conservation.

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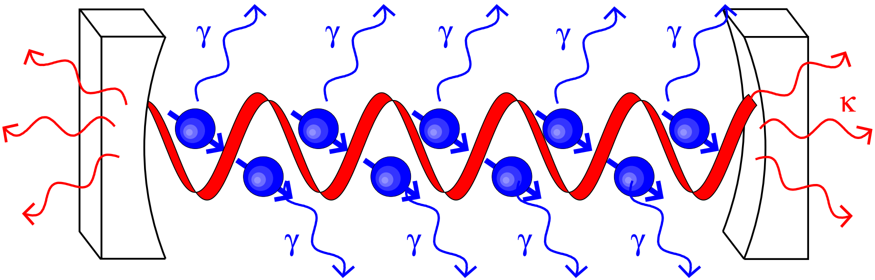

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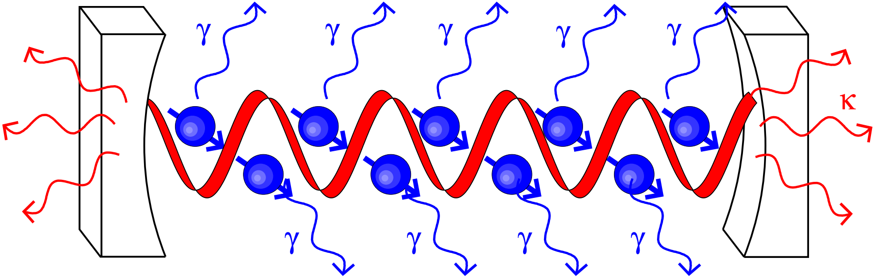

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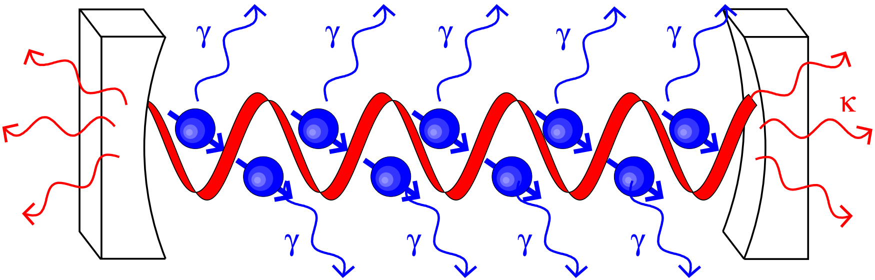

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MFT/Maxwell-Bloch: limiting cases

Denote: $\alpha = \langle \hat{a} \rangle$, $s^i = \langle \sigma^i \rangle$.

$$\begin{aligned}
\partial_t \alpha &= -i\omega\alpha - igNs^x - \kappa\alpha/2 \\
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Wigner function $W(\hat{a} = x + ip)$,

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  - Superradiance: bimodal state
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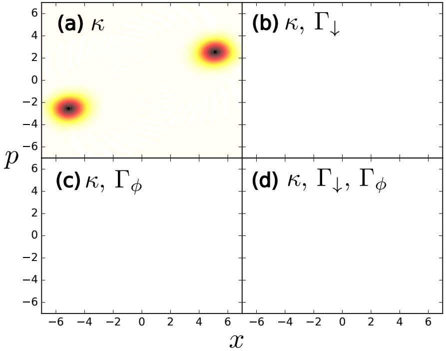

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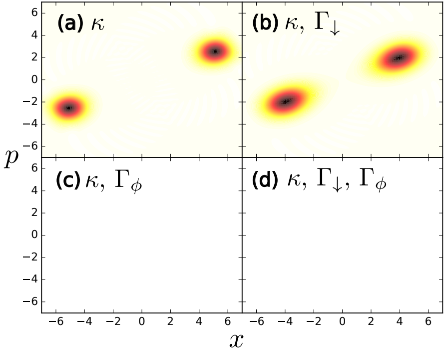

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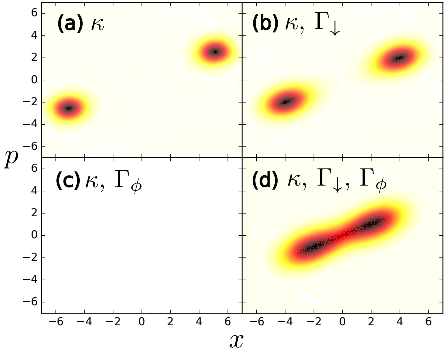

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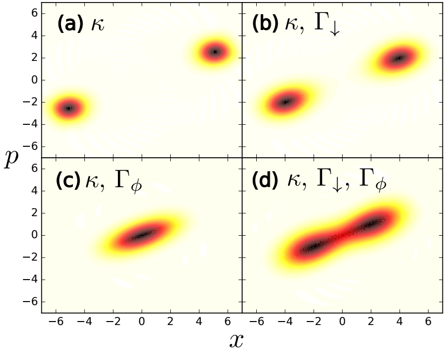

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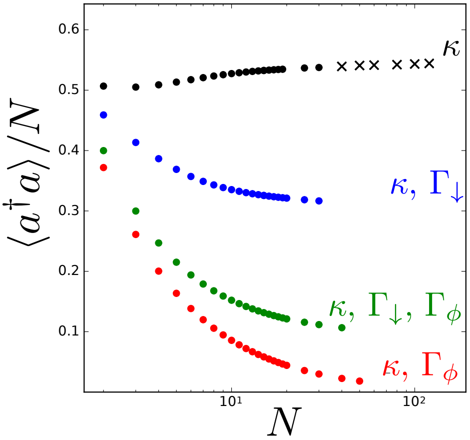

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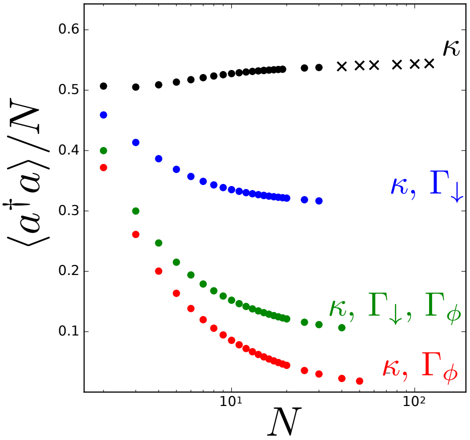

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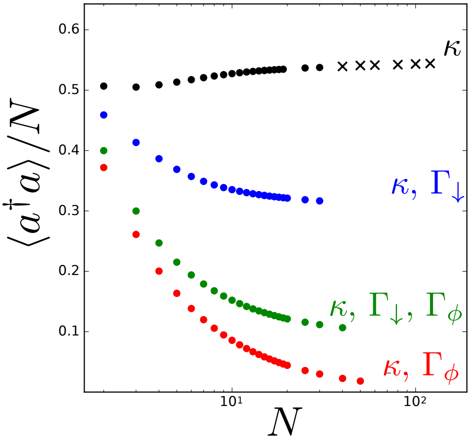

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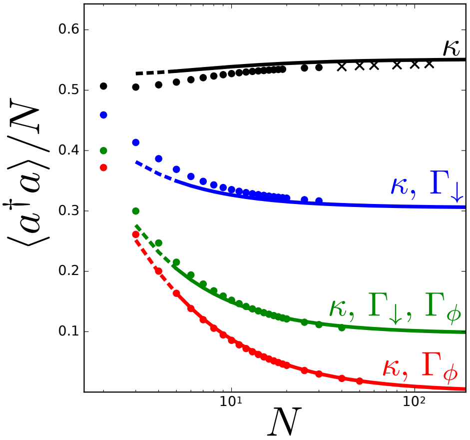

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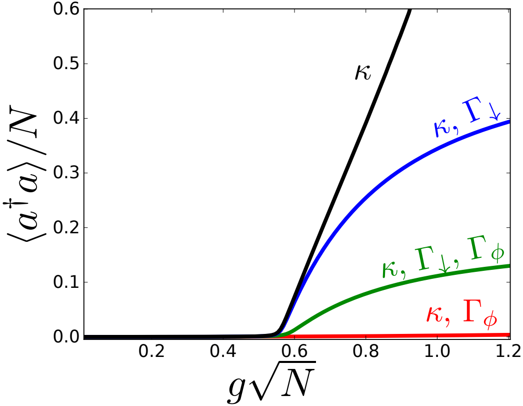
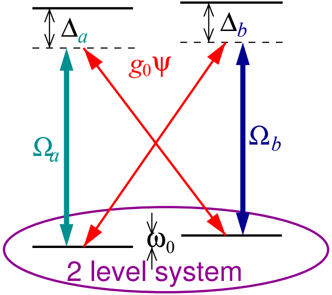


Raman driven Dicke model

- Imbalanced Hamiltonian, from above:

$$H = \omega_0 S^z + g(\hat{a}S^+ + \hat{a}^\dagger S^-) + g'(\hat{a}S^- + \hat{a}^\dagger S^+) + \omega \hat{a}^\dagger \hat{a} + U\hat{a}^\dagger \hat{a} S^z$$



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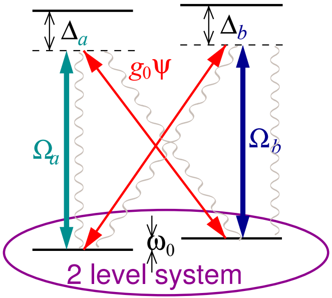

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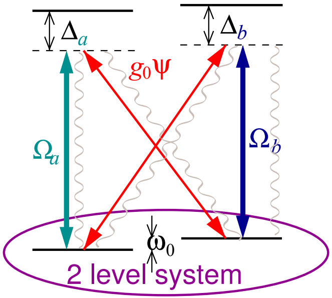

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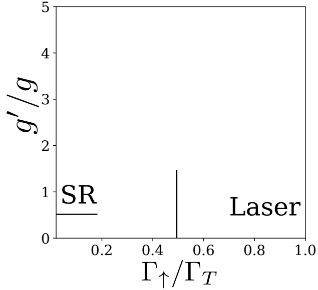
Lasing and superradiance phase diagram

- Cumulant calculation. ($\Gamma_T = \Gamma_\uparrow + \Gamma_\downarrow$)
- Mean field stability phase-diagram — # unstable modes



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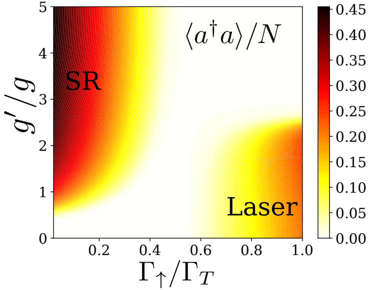

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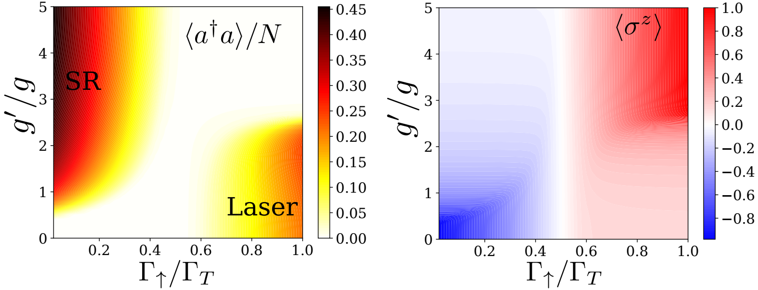

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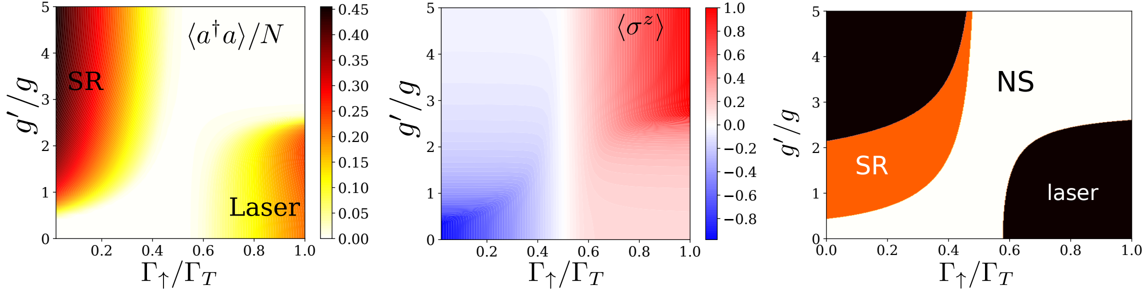
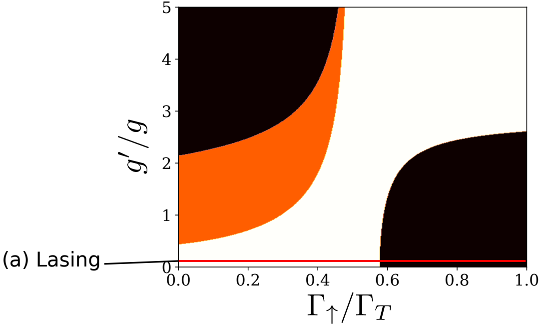
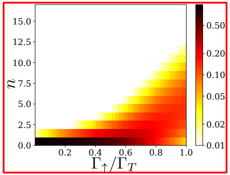


Signatures of states: Photon number distribution

- Can SR/Lasing be distinguished by photons?
- Coherent state — similar $P(n)$





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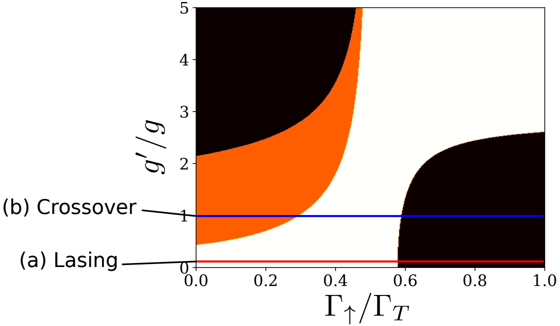
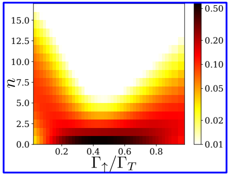
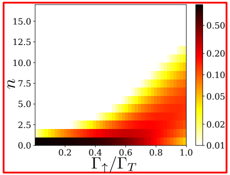
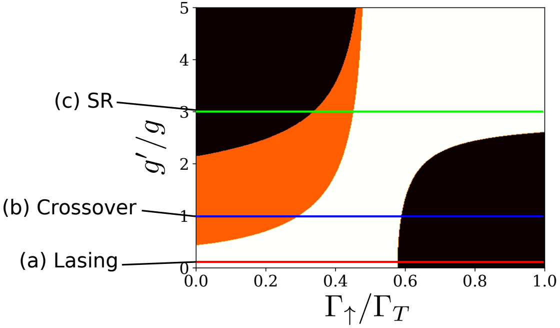


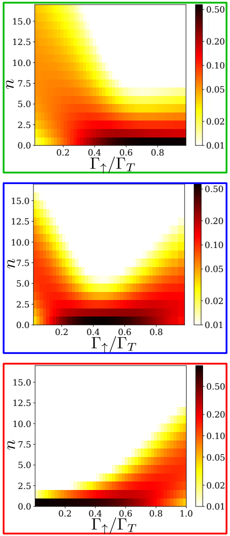




Signatures of states: Photon number distribution

- Can SR/Lasing be distinguished by photons?
- Coherent state — similar $P(n)$





Signatures of states: Fluoresence spectrum

- Fluoresence/emission — driven system

  $$S(\nu) = \int_{-\infty}^{\infty} \left\langle \hat{a}^\dagger(t)\hat{a}(0) \right\rangle e^{i\nu t}\, dt$$

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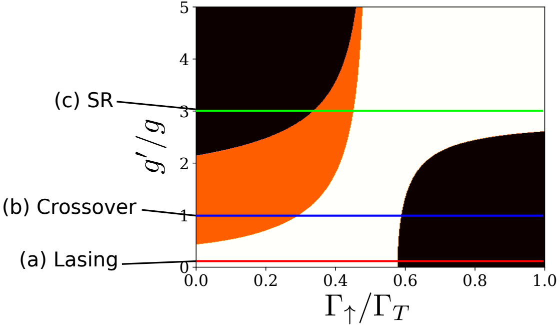

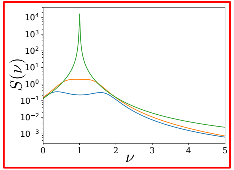

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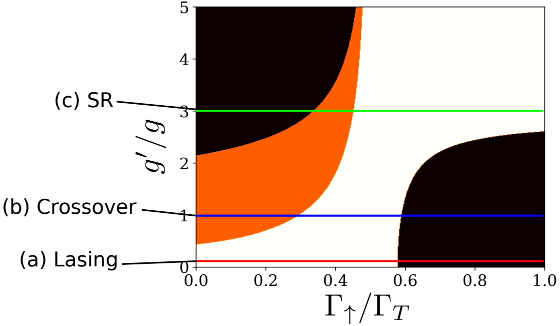

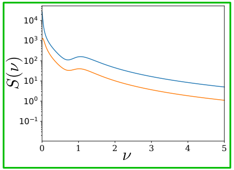

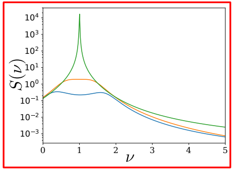

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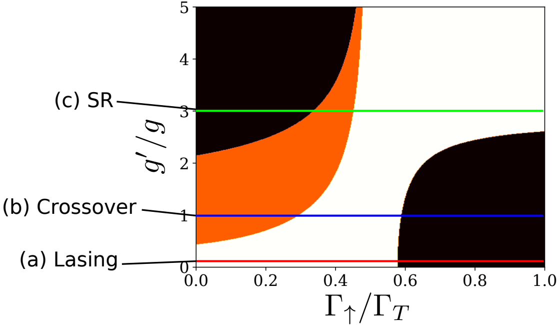

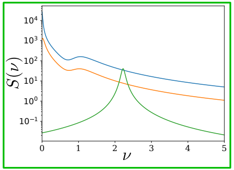

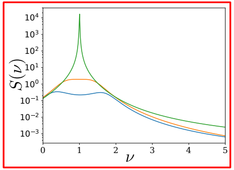

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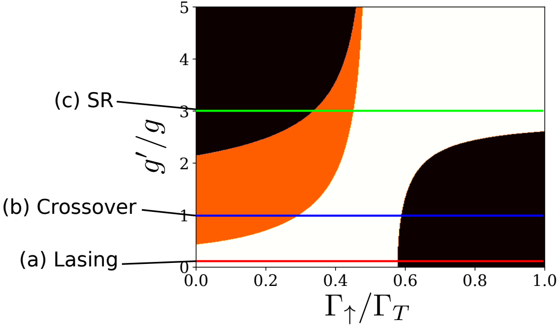

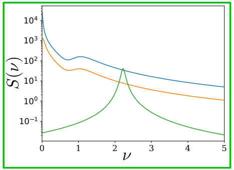

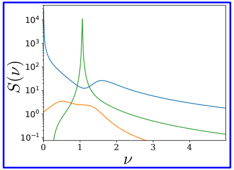

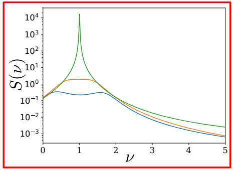

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1. Introduction: Open Dicke model reminder
   - Dicke superradiance vs lasing
   - Driven Dicke model
2. Behaviour with dephasing
   - Mean field theory problem
   - Exact solution and cumulant expansion
3. Lasing vs superradiance: competition
   - Basic phase diagram
   - Signatures of states
   - Blue detuned pump

Blue detuned pumping photon

- Large $\Gamma_\uparrow$ phase similar to inverted SR
- Small $\Gamma_\uparrow$ phase inverted laser
- Speckle/noise — absence of steady state

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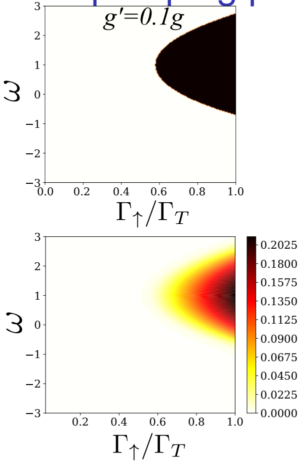

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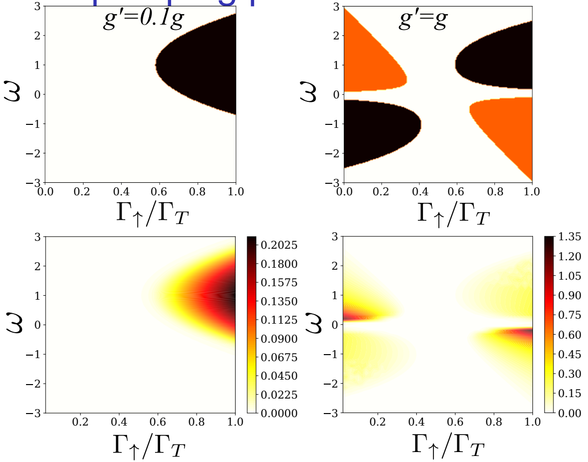

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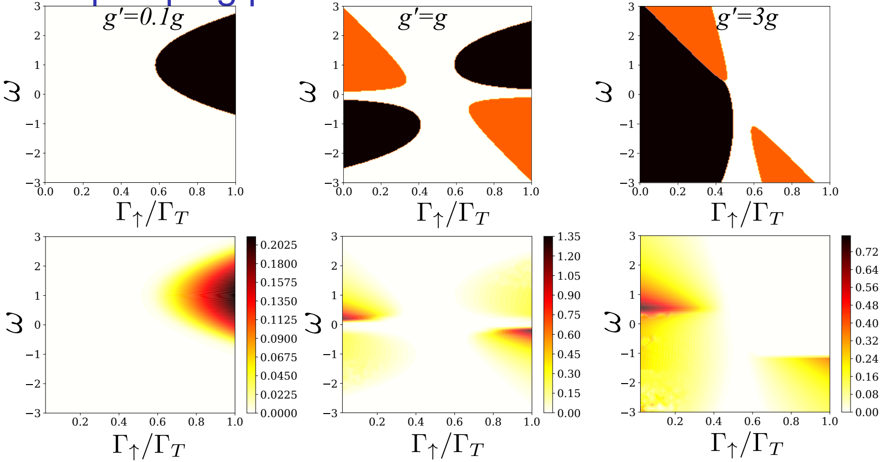

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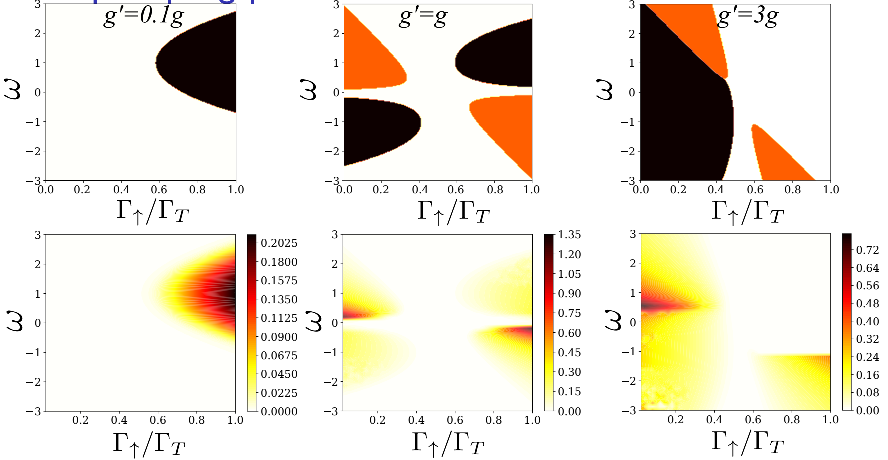

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Chaotic dynamics

- MF bifurcation diagram: Im(α) at Re$(\alpha) = 0$, Re$(\dot{\alpha}) > 0$ ($g' = 2.3g$)

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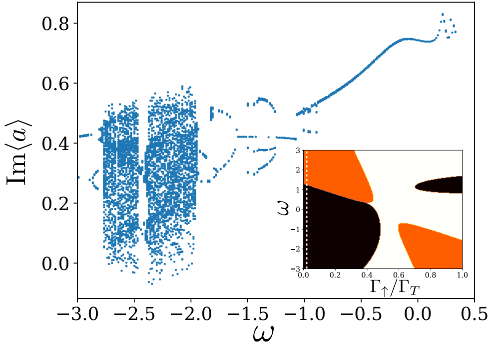

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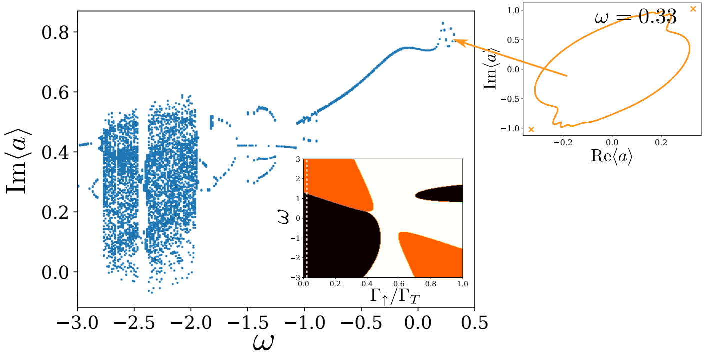

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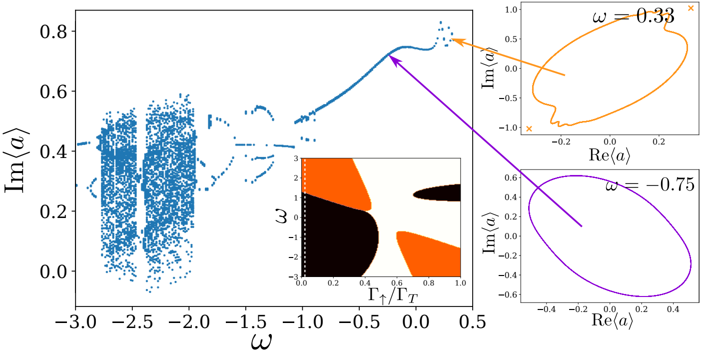

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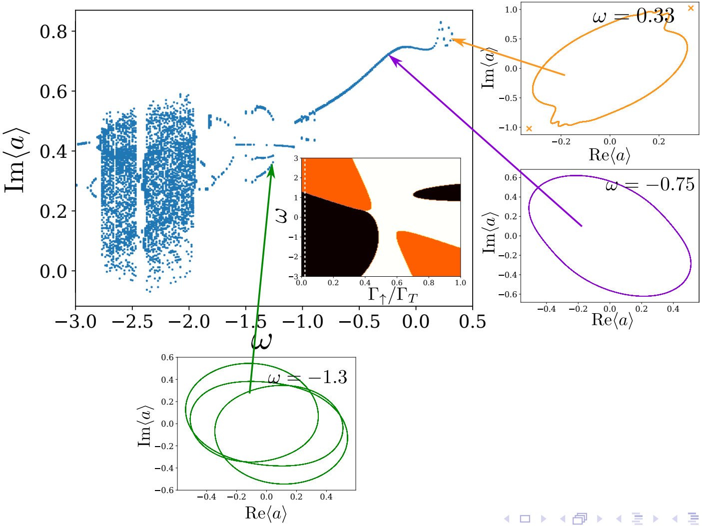

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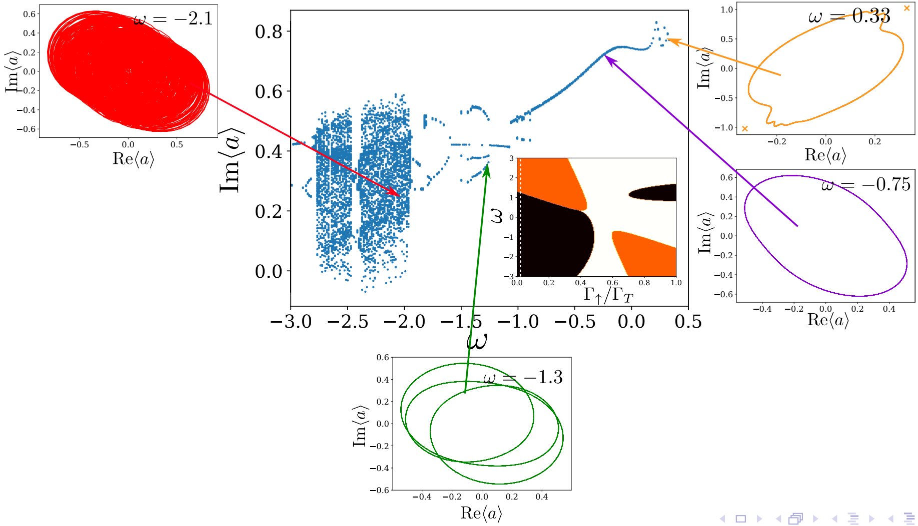

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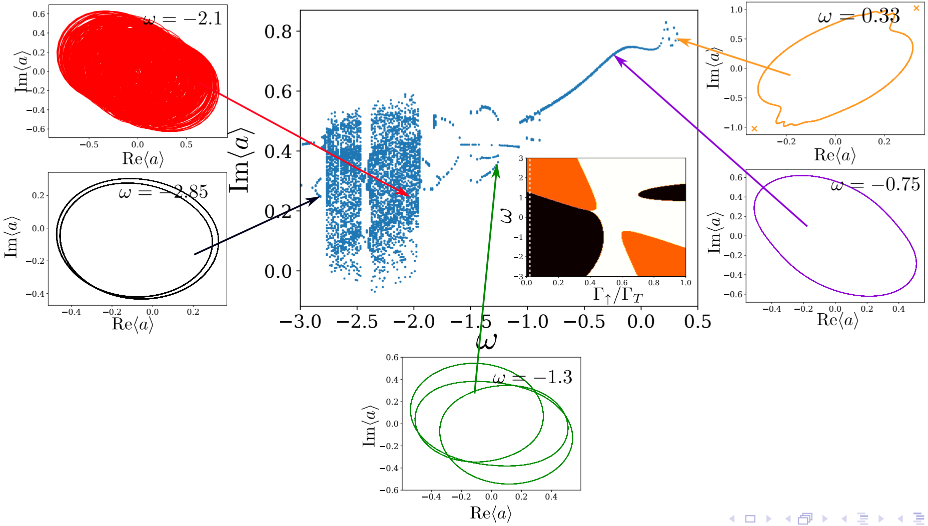

Acknowledgements

GROUP:



FUNDING:

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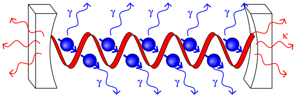
Engineering and Physical Sciences Research Council

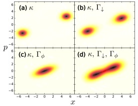
topnes
Topological Protection and Non-Equilibrium States in Strongly Correlated Electron Systems

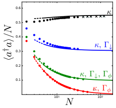
Hybrid Polaritonics

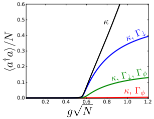
Summary

- Open Dicke model, $\kappa, \Gamma_\phi, \Gamma_\downarrow$, Exact numerics & cumulants



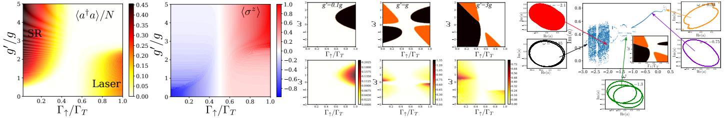


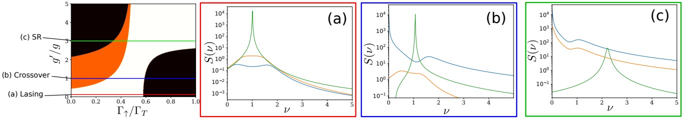




[Kirton & JK, PRL '17]

- Laser/Dicke crossover model





[Kirton & JK, arXiv:1710.06212]

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- **But:** Permutation symmetry of ρ remains

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- Green's function: Response to weak perturbation

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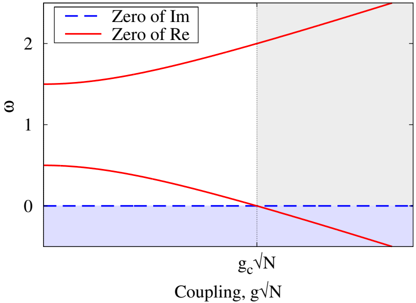

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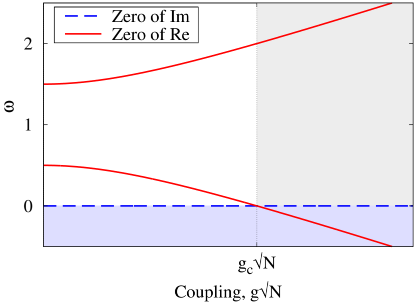
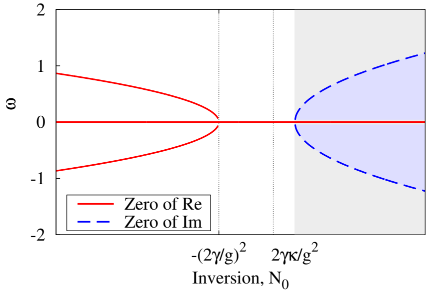


Laser



Bosons beyond Dicke — single mode

So far $\Psi(\mathbf{r}) = \chi_0 + \chi_1 2\cos(qx)\cos(qz) \longrightarrow \mathbf{S} = \chi^\dagger \sigma \chi$.
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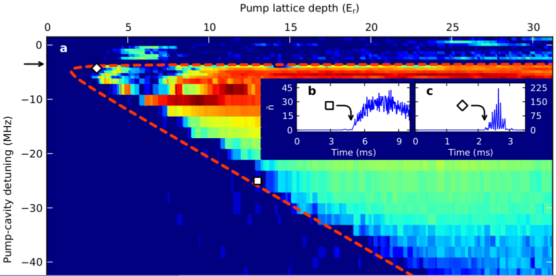

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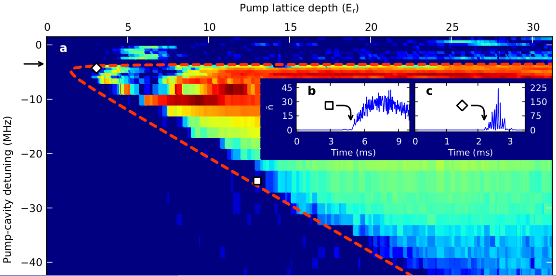

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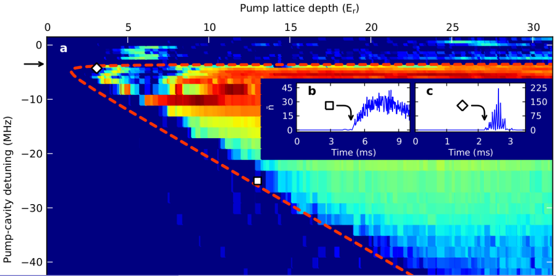

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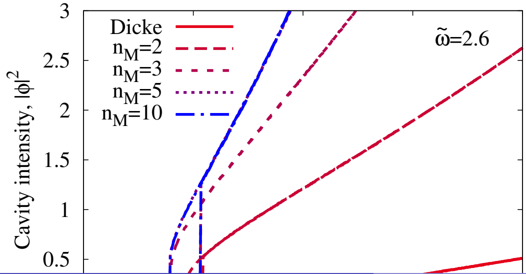
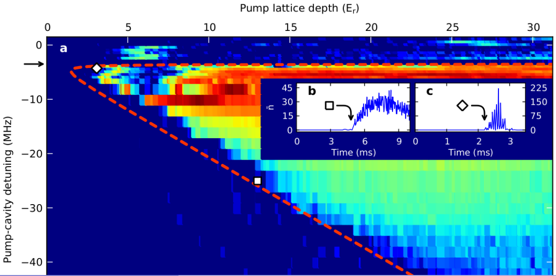
Truncate $|\mathbf{n}| < n_M$ — Hysteresis at intermediate ω





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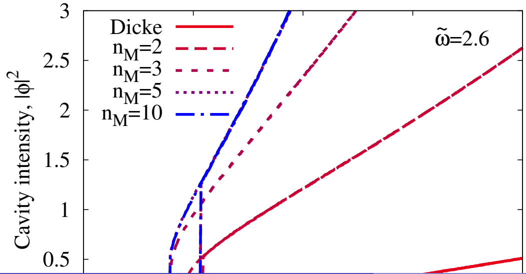
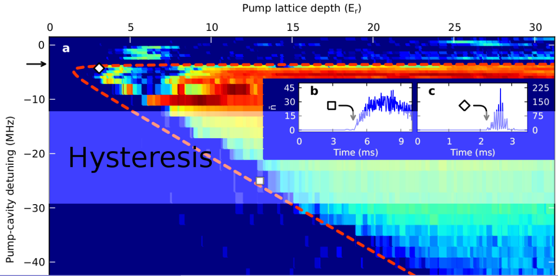
Generally $\Psi(\mathbf{r}) = \sum_{\mathbf{n}} \chi_{\mathbf{n}} e^{i q \mathbf{n}\cdot r}$.

$$i\partial_t \chi_{\mathbf{n}} = \omega_r \left(|\mathbf{n}|^2 \delta_{\mathbf{n},\mathbf{n}'} - V_{\mathbf{n},\mathbf{n}'}(\alpha) \right) \chi_{\mathbf{n}}$$

$$i\partial_t \alpha = (\omega - E_0 \sum_{\mathbf{n},\mathbf{n}'} \chi^*_{\mathbf{n}} m^{(2)}_{\mathbf{n},\mathbf{n}'} \chi_{\mathbf{n}'} - i\kappa)\alpha - \eta E_0 \sum_{\mathbf{n},\mathbf{n}'} \chi^*_{\mathbf{n}} m^{(1)}_{\mathbf{n},\mathbf{n}'} \chi_{\mathbf{n}'}$$

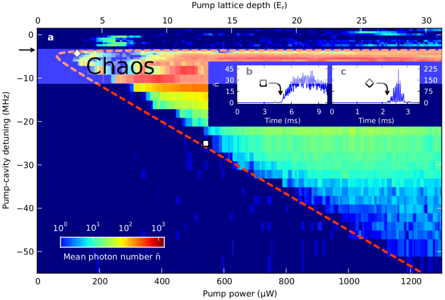
Truncate $|\mathbf{n}| < n_M$ — Hysteresis at intermediate ω





Bosons beyond Dicke — chaos

Near resonance: irregular dynamics
(NB $\omega_{\text{Pump}} - \omega_{\text{cavity}} = -\omega$)



Bosons beyond Dicke — chaos

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