Superradiance and self-organisation of cold atoms in optical cavities

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Coupling many atoms to light

Old question: *What happens to radiation when many atoms interact "collectively" with light.*

Superradiance — dynamical and steady state.

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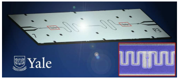
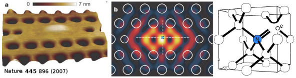
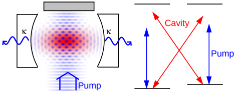
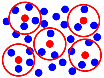
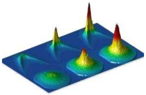
New relevance

- Superconducting qubits 

- Quantum dots & NV centres 

- Ultra-cold atoms 

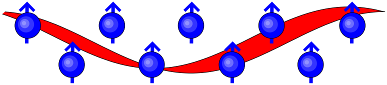
- Rydberg atoms/polaritons 
- Microcavity Polaritons 

Dicke effect: Superradiance

PHYSICAL REVIEW VOLUME 93, NUMBER 1 JANUARY 1, 1954

Coherence in Spontaneous Radiation Processes

R. H. Dicke
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey



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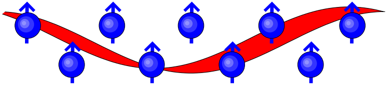

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Collective decay:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+S^-\rho - S^-\rho S^+ + \rho S^+S^-\right]$$

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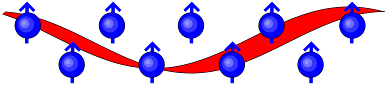

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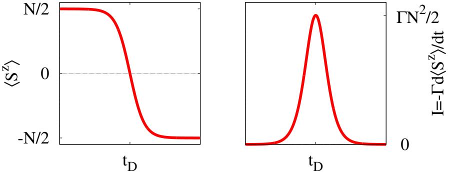

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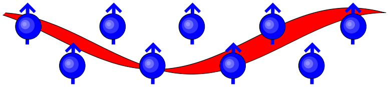

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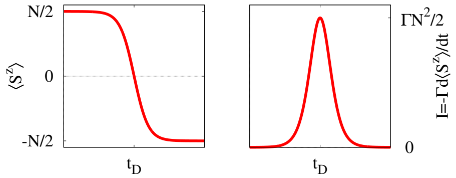
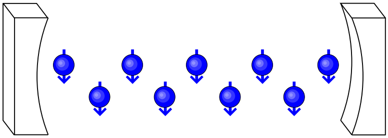


Problem: dipole interactions dephase. [Friedberg et al, Phys. Lett. 1972]

Dicke model and Dicke-Hepp-Lieb transition



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[Hepp, Lieb, Ann. Phys. '73]

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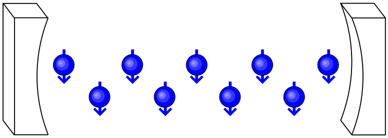

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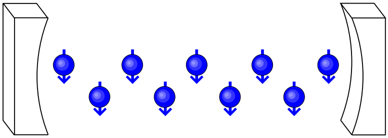

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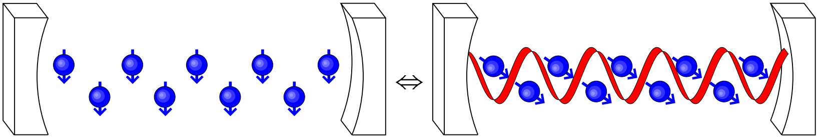

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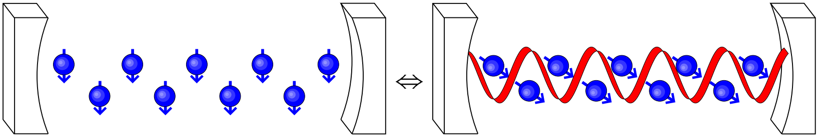

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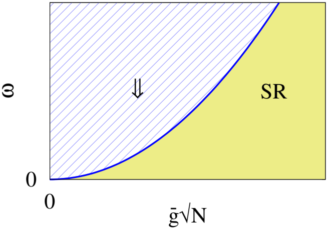

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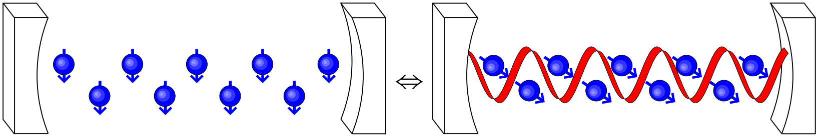
No go theorem for Dicke-Hepp-Lieb transition



Spontaneous polarisation if: $4Ng^2 > \omega\omega_0$

[Rzazewski *et al* PRL '75]

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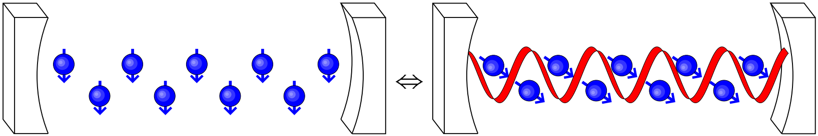

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No go theorem:. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \Leftrightarrow g(\psi^\dagger + \psi)(S^- + S^+), \qquad \sum_i \frac{A^2}{2m} \Leftrightarrow N\zeta(\psi + \psi^\dagger)^2$$

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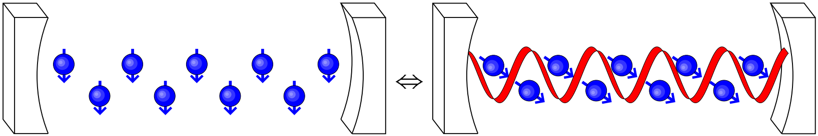

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For large N, $\omega \to \omega + 4N\zeta$. (RWA)

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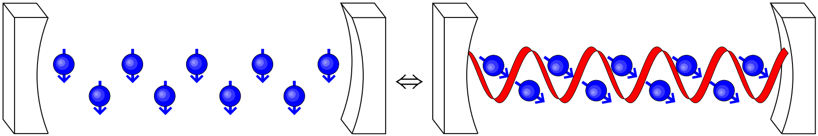

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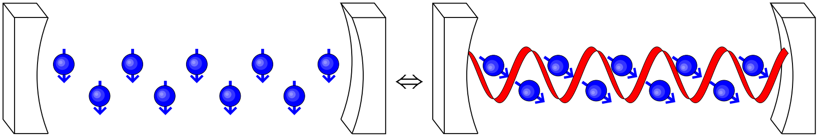

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But f-sum rule states: $g^2/\omega_0 < \zeta$. **No transition**

[Rzazewski *et al* PRL '75]

Ways around the no-go theorem

Problem: $g^2/\omega_0 < \zeta$ for intrinsic parameters. **Solutions**:

1. Ferroelectric transition in $\mathbf{D} \cdot \mathbf{r}$ gauge.
   [JK JPCM '07, Vukics & Domokos PRA 2012]
   - Circuit QED [Nataf and Ciuti, Nat. Comm. '10; Viehmann *et al.* PRL '11]
2. Grand canonical ensemble:
   - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only:
     $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
   - Incoherent pumping — polariton condensation.
3. Dissociate g, ω_0.
   e.g. Raman scheme: $\omega_0 \ll \omega$.
   [Dimer *et al.* PRA '07; Baumann *et al.* Nature '10. Also, Black *et al.* PRL '03]

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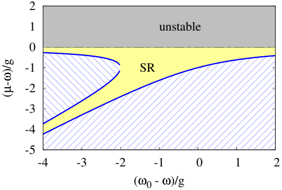

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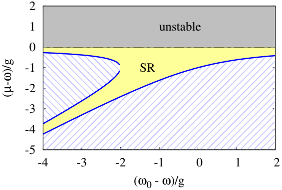

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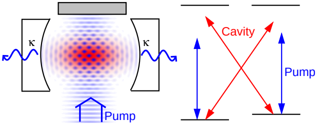

Outline

1. Dicke model, superradiance and no–go theorem
2. Superradiance and self-organisation
   - Raman scheme
   - Hierarchies of approximation
   - Equilibrium theory of Dicke
3. Fermionic self organisation
   - Equilibrium phase diagrams
   - Landau theory and microscopics
   - Open system?
4. Open system dynamics of Bososn
   - Attractors of open Dicke model
   - Bosons beyond Dicke
5. Conclusions

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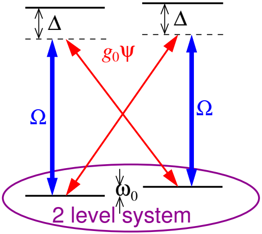

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- Imbalanced case (internal states):
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- New “feedback” term $U = \dfrac{g_0^2}{2\Delta_b} - \dfrac{g_0^2}{2\Delta_a}$

[Dimer *et al.* PRA ’07]

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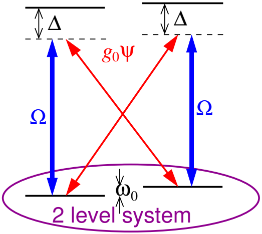

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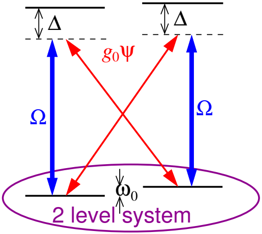

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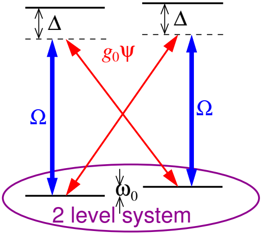

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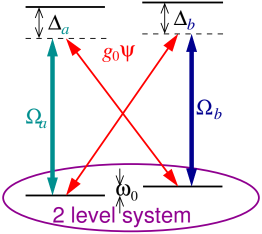

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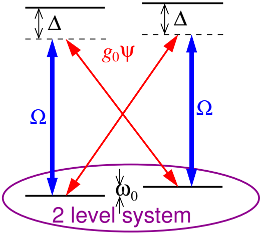

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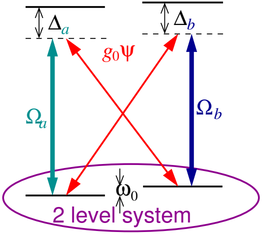
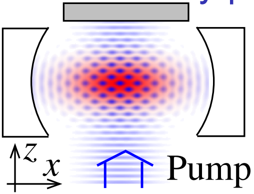


[Dimer *et al.* PRA '07]

Transversely pumped cavity



Internal state → momentum states

1. Full description

$$H_0 = \omega_{\text{cavity}}\psi^\dagger\psi + \int d^2r \Big[\sum_{\alpha=e,g} c^\dagger_\alpha \left(\frac{-\nabla^2}{2m}\right) c_\alpha$$

$$+ \omega_{\text{atom}} c^\dagger_e c_e + E(x,z,t)(c^\dagger_e c_g + c^\dagger_g c_e) \Big]$$

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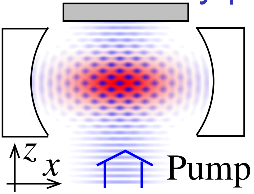

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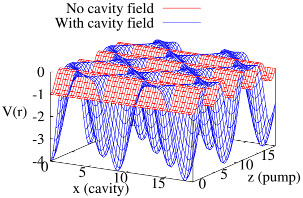
2. Eliminate e state
   - Rotating frame $\omega = \omega_{\text{cavity}} - \omega_{\text{pump}} - N\delta$



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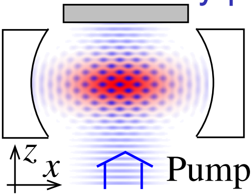

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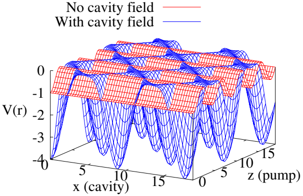

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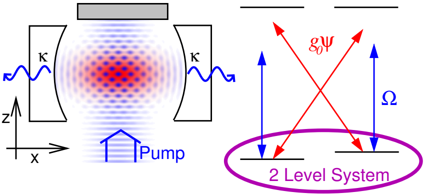
3. Dicke: project to atomic states $\phi(x,z) \propto \begin{cases} 1 \\ \cos(qz)\cos(qz) \end{cases}$

Mapping transverse pumping to Dicke model



Reduced basis:

$$\phi(x,z) \propto \begin{cases} 1 & \Downarrow \\ \cos(qz)\cos(qz) & \Uparrow \end{cases}$$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+)$$

[Baumann *et al* Nature '10]

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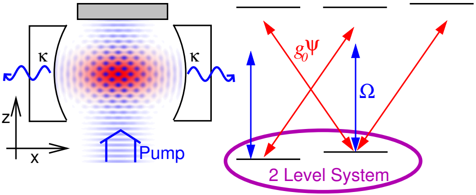

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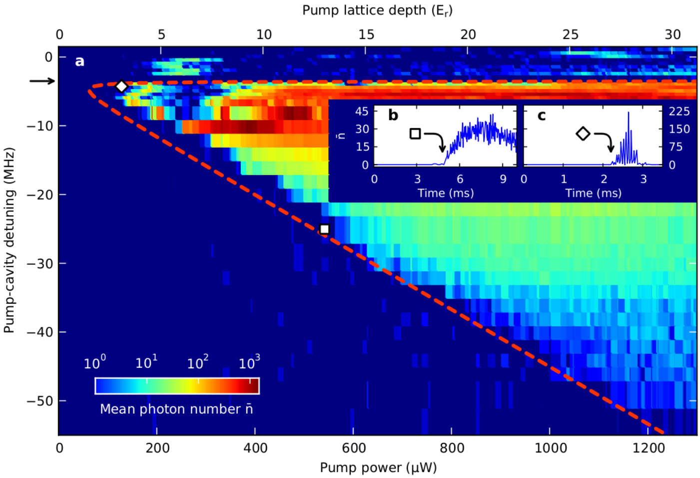
"Feedback" due to extra states $U = -\dfrac{g_0^2}{4\Delta}$

[Baumann *et al* Nature '10]

Experimental phase diagram



- Pump power $g \propto \sqrt{\text{Power}}$
- Pump-cavity detuning $\omega \sim -\Delta$

[Baumann *et al* Nature '10]

Phase diagram of extended Dicke model

Ground state energy, $\lambda = \langle \psi \rangle / \sqrt{N}$:

$$\frac{E}{N} = \omega \lambda^2 - \frac{N}{2} \sqrt{(\omega_0 + UN\lambda^2)^2 + 16 g^2 N \lambda^2}$$

- Superradiant transition:

$$4g^2 N > \left(\omega - \frac{UN}{2} \right) \omega_0$$

- Stability, $\lambda \to \infty$

$$E \sim \left(\omega - \frac{U N}{2} \right) \lambda^2 + \ldots$$

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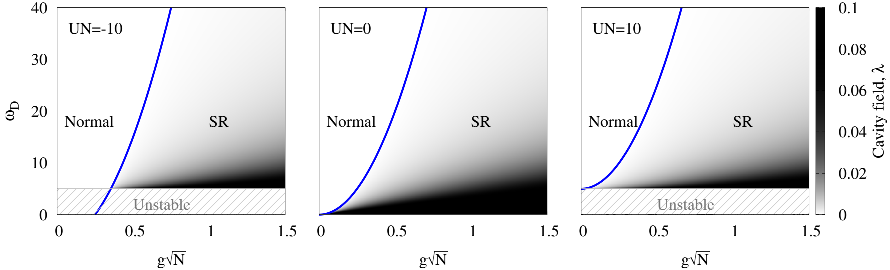
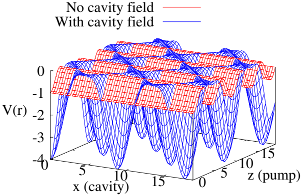


Outline

Fermions in optical cavities



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[Domokos & Ritsch, PRL '03; Black *et al.* PRL '03; Piazza, Strack, Zwerger, Ann. Phys '13]

- Pauli blocking
- Commensurability effects

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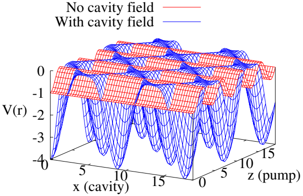

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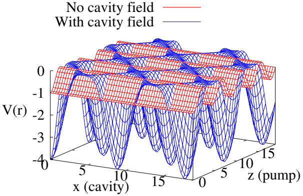

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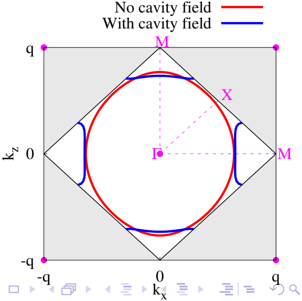

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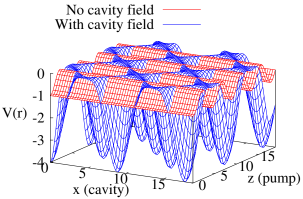

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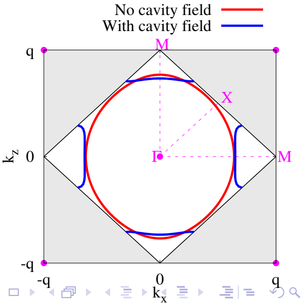
[JK, Bhaseen, & Simons; Piazza & Strack; Chen *et al.* All PRL '14.]



Dimensionless variables and free energy

- Rescale with $\sqrt{2}q, \omega_r = \hbar^2 q^2/2m$, Dimensionless variables:
  - $N/N_L = n_F$
  - $\omega \to \tilde{\omega}$
  - $\Omega \to \eta$
  - $\langle\psi\rangle \to \phi$
- Free energy $f = F/N_L\omega_r$

$$f(\tilde{\omega}, \eta, n_F \to \mu; \phi) = \tilde{\omega}\phi^2 + \mu n_F - \frac{1}{\beta}\int_{BZ} d^2k \sum_n \ln\left[1 + e^{-\beta(\epsilon_{\mathbf{k},n} - \mu)}\right]$$

- $\epsilon_{\mathbf{k},n}$ from $\hat{h} = -\nabla^2 - V(\eta, \phi; \mathbf{r})$
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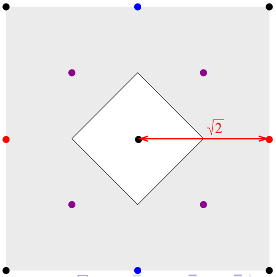

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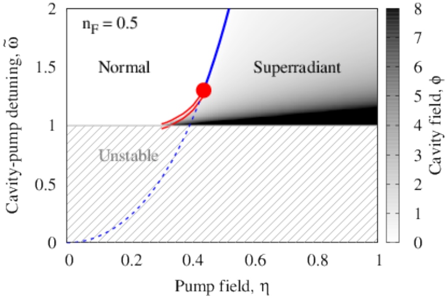

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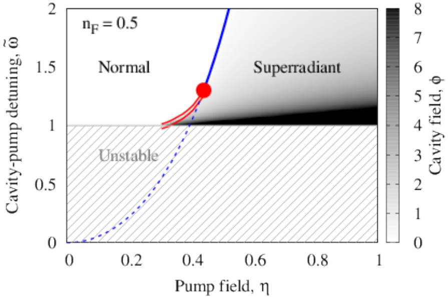

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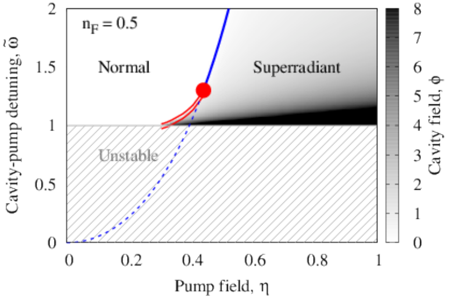

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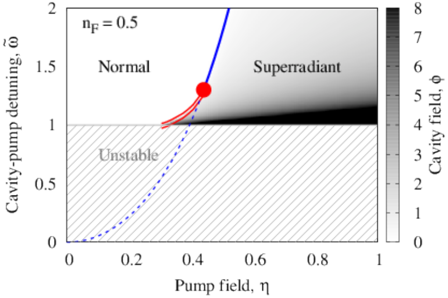
$$f = a\phi^2 + b\phi^4 + c\phi^6$$

$b < 0$ at small η.

Origin of first order transition



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- Second order perturbation theory, $-\phi^4|m_{\mathbf{k},\mathbf{k}'}|^2/(E_{\mathbf{k}'} - E_{\mathbf{k}})$
- Larkin-Pikin like mechanism
- Survives to low n_F: Bosons!
  - But needs state $\phi(x, z) = \cos(\sqrt{2}x)$
  - Missed by Dicke model

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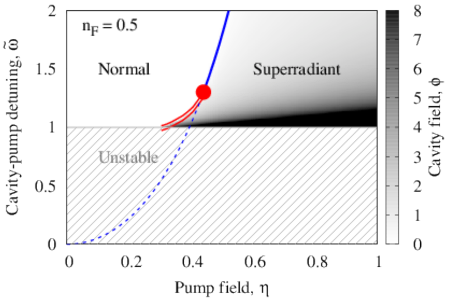

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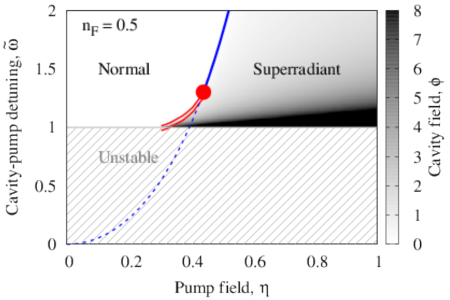

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- Larkin-Pikin like mechanism
- Survives to low n_F: Bosons!
  - But needs state $\phi(x,z) = \cos(\sqrt{2}x)$
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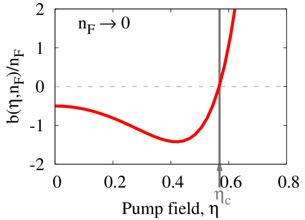
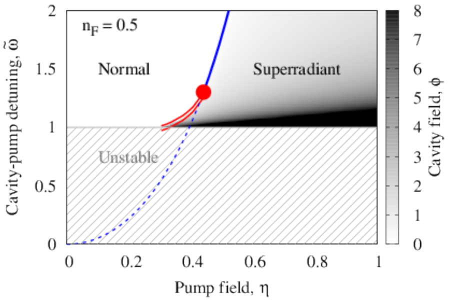


Origin of first order transition



- $\epsilon_{\mathbf{k},n}$ from $\hat{h} = k^2\delta_{\mathbf{k},\mathbf{k}'} - V_{\mathbf{k},\mathbf{k}'}$

$$V_{\mathbf{k},\mathbf{k}'} = \phi^2 \sum_s \delta_{\mathbf{k},\mathbf{k}'+s\sqrt{2}\hat{\mathbf{x}}} + \eta\phi \sum_{s,s'} \delta_{\mathbf{k},\mathbf{k}'+\frac{s}{\sqrt{2}}\hat{\mathbf{x}}+\frac{s'}{\sqrt{2}}\hat{\mathbf{z}}} + \eta^2 \sum_s \delta_{\mathbf{k},\mathbf{k}'+s\sqrt{2}\hat{\mathbf{z}}}$$

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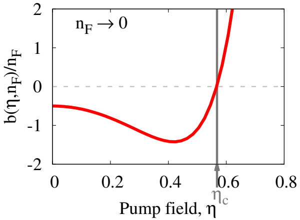

Higher fillings

$$f = a\phi^2 + b\phi^4 + c\phi^6$$

- Phase diagram unchanged for $n_F < 1$
- 2nd order line $a = 0$
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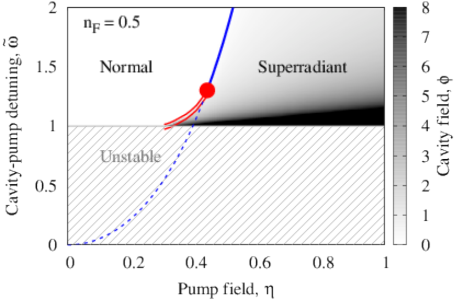

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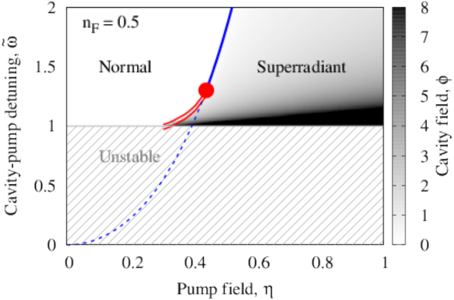

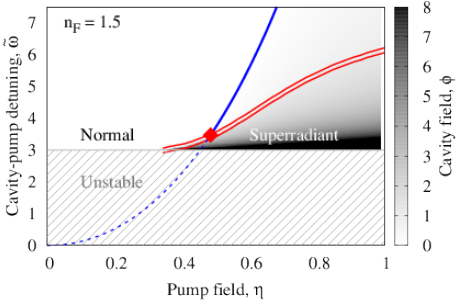

- 2nd band, new structure.
  - Critical end-point ◆
  - $a = 0$ line cut by 1st order
- SR–SR phase boundary
  - No symmetry breaking
  - Liquid–gas type (metamagnetic)

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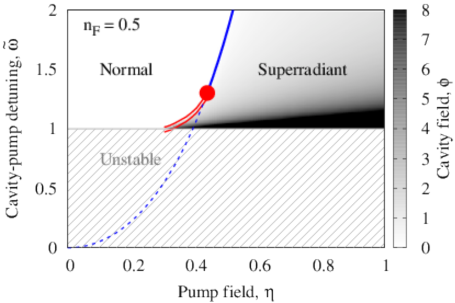

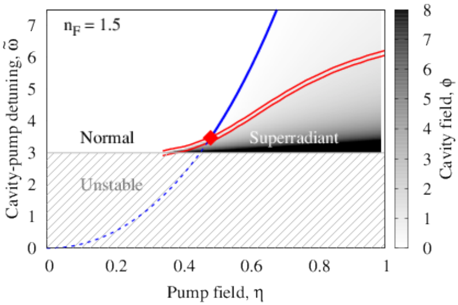

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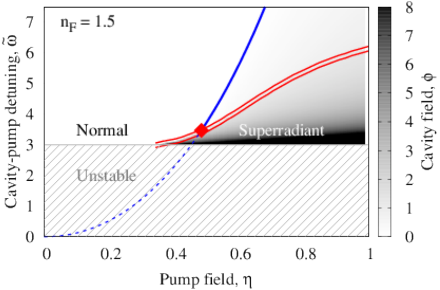

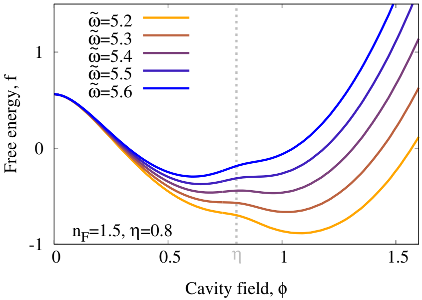

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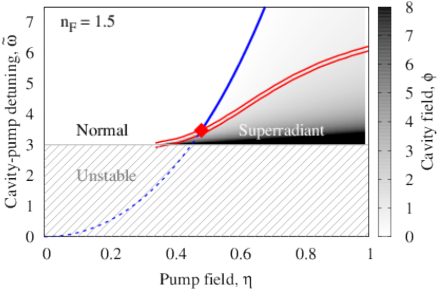

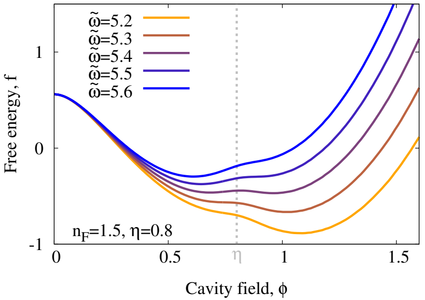

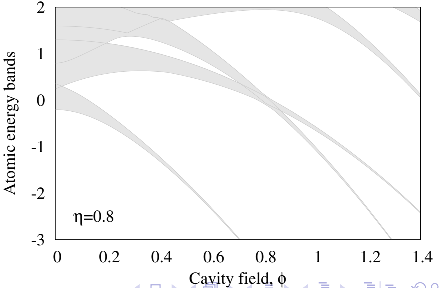

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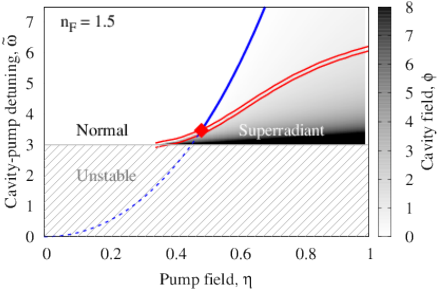

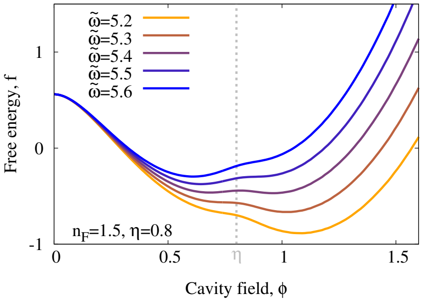

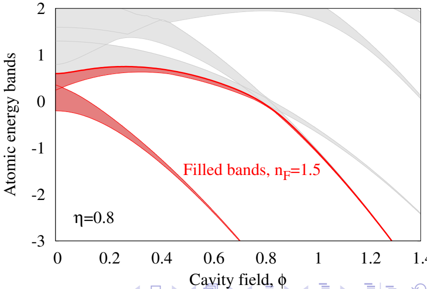

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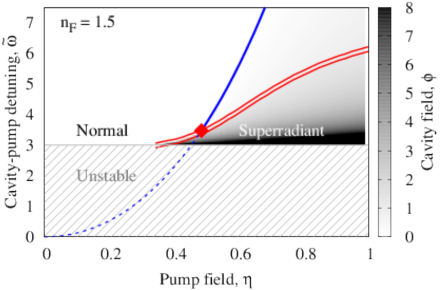

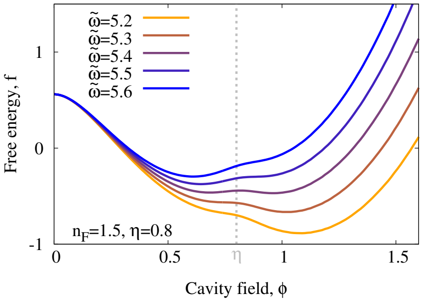

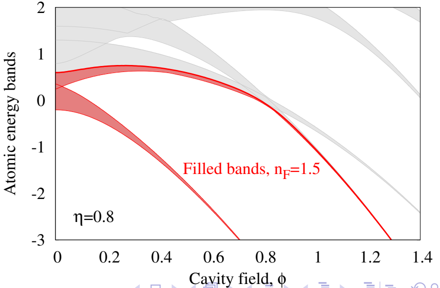

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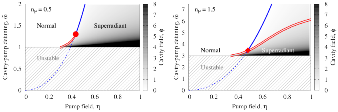
Phase diagram vs density

- Phase topology change:



- Fix η, plot vs n_F
- SR–SR after critical point

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  Susceptibility χ asymptote $\eta \to \infty$

$$\chi \simeq 16\eta^2 \ln\left|\frac{1 - n_F}{1 + nf}\right|$$

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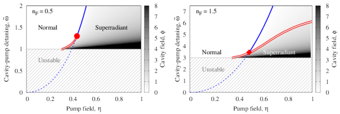

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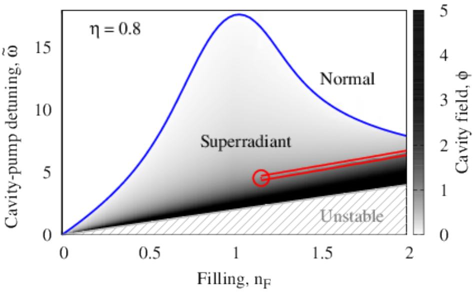

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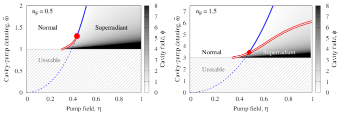

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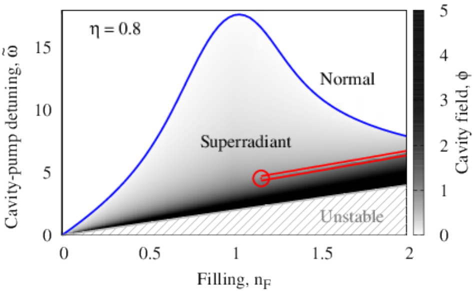

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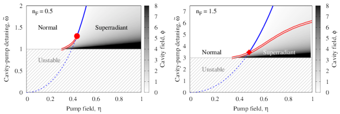

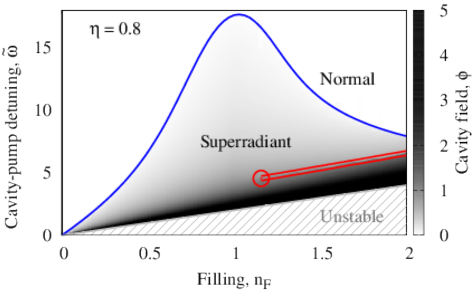

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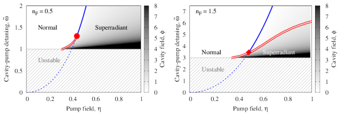

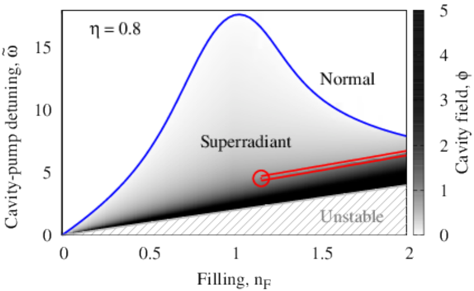

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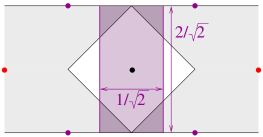

Open system vs ground state phase diagram

- Open system, $\dot{\rho} = -i[H, \rho] - \kappa\mathcal{L}[\psi]$. Stable attractors
- What survives — Normal-SR boundary
  - Fluctuations $\delta\phi = ue^{-i\nu t} + v^* e^{i\nu^* t}$.
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  - Unstable region → new attractors
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$$\frac{\tilde{\omega}^2 + \tilde{\kappa}^2}{\tilde{\omega}} = -\chi(\eta, n_F)$$

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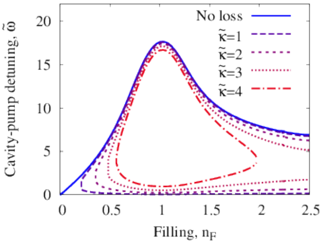

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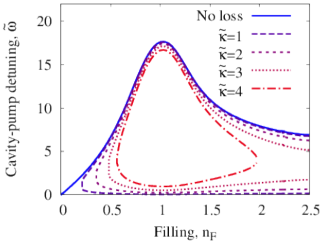

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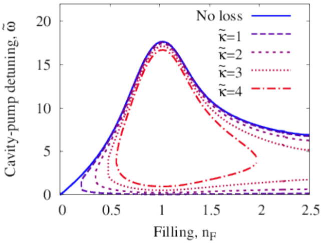

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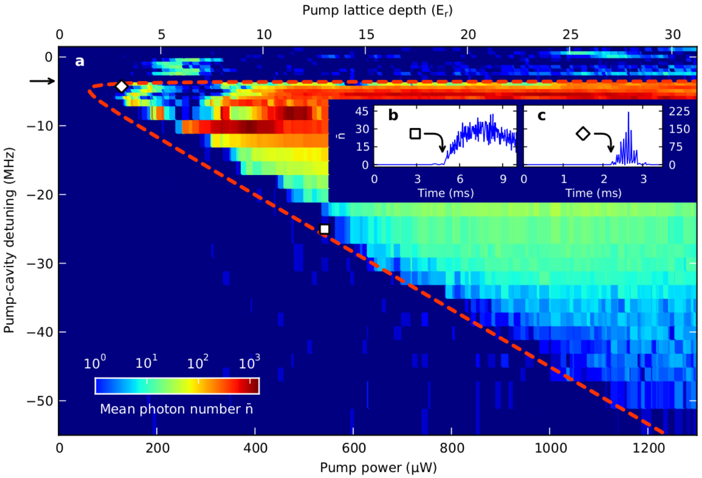
Outline

Experimental phase diagram



- Pump power $g \propto \sqrt{\text{Power}}$
- Pump-cavity detuning $\omega \sim -\Delta$

[Baumann *et al* Nature '10]

Dicke model classical dynamics

Open dynamical system:

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + U S_z \psi^\dagger\psi.$$

$$\partial_t \rho = -i[H, \rho] - \kappa(\psi^\dagger\psi\rho - 2\psi\rho\psi^\dagger + \rho\psi^\dagger\psi)$$

- Fixed points: $\dot{S} = 0, \dot{\psi} = 0$
- Limit cycles?

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Classical EOM
($|\mathbf{S}| = N/2 \gg 1$)

$$\dot{S}^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^*)S^z$$

$$\dot{S}^z = ig(\psi + \psi^*)(S^- - S^+)$$

$$\dot{\psi} = -[\kappa + i(\omega + US^z)]\psi - ig(S^- + S^+)$$

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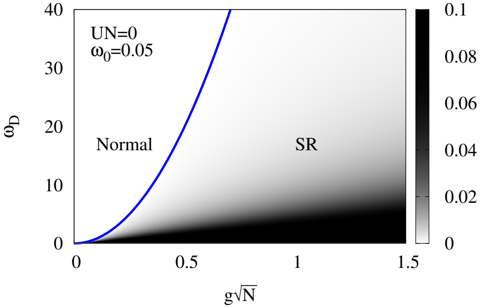
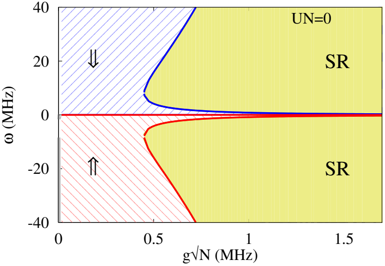
Long-time behaviour:

- Fixed points: $\dot{\mathbf{S}} = 0, \dot{\psi} = 0$
- Limit cycles?

Equilibrium Dicke vs open phase diagram, $UN = 0$





- Shift boundary $(\kappa^2 + \omega^2)/\omega = -\chi(\omega_0)$
- Allow negative $\omega \to$ inverted

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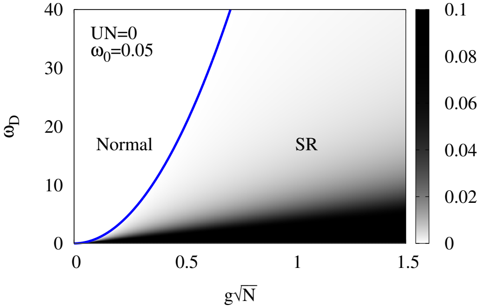

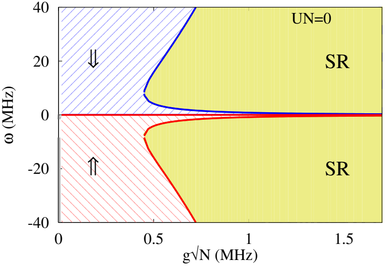

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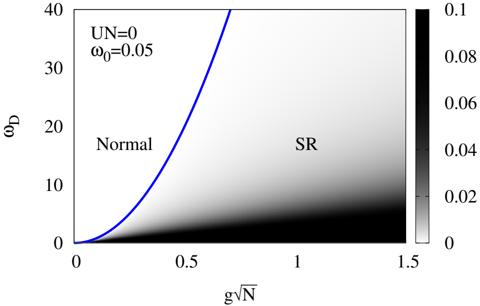

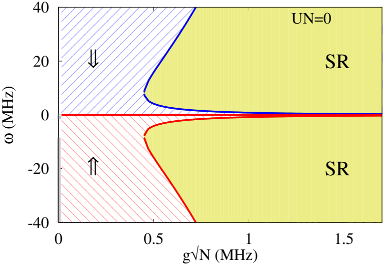

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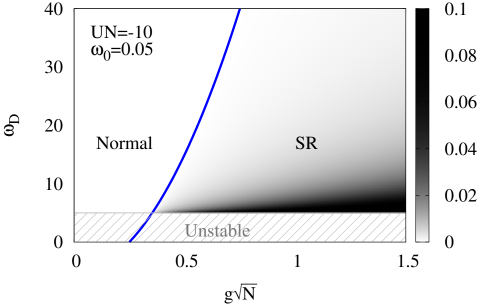
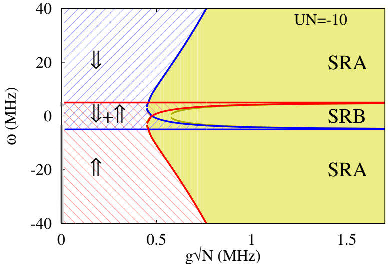
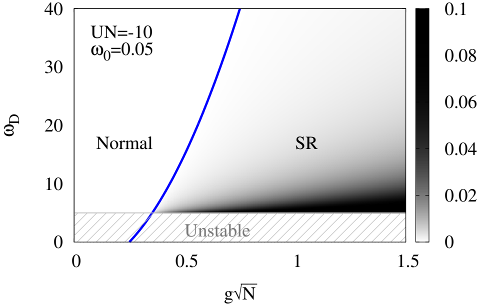
. . . Dicke . . . $UN = -10$MHz

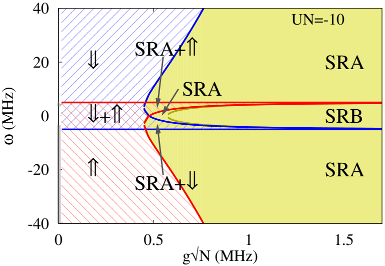




- Coexistence regions
- Unstable → SRB

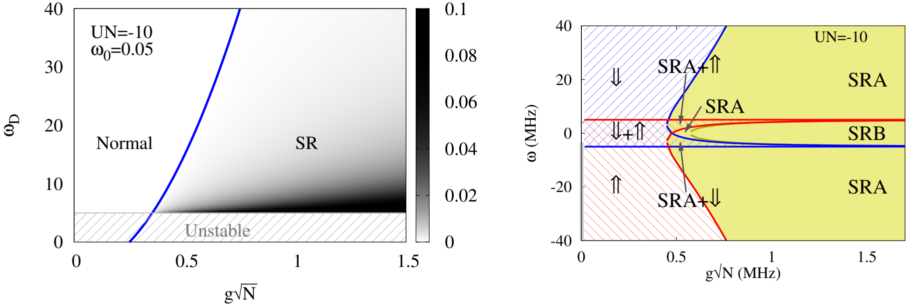
. . . Dicke . . . $UN = -10$MHz





- Coexistence regions

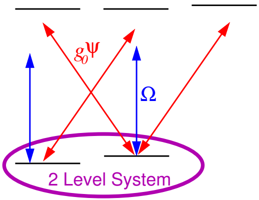
…Dicke … $UN = -10$MHz

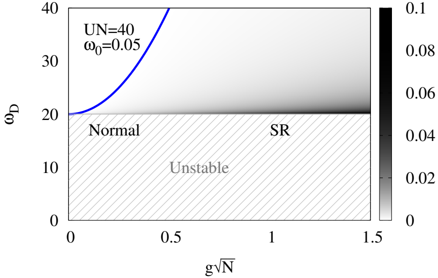


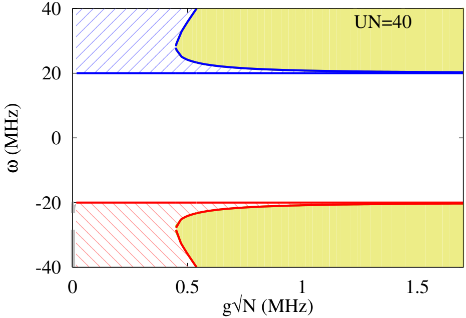
- Coexistence regions
- Unstable → SRB

. . . Dicke . . . $UN = +40$MHz

Changing U:







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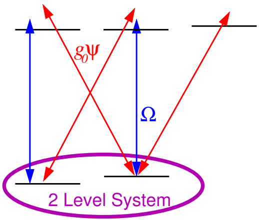

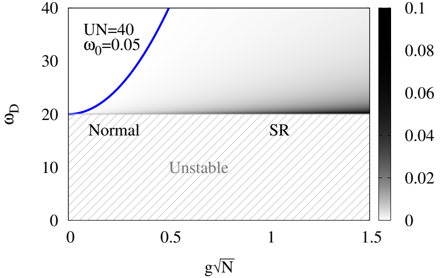

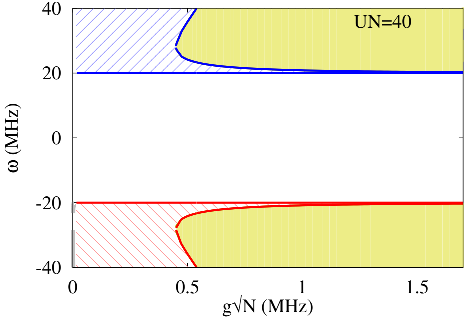

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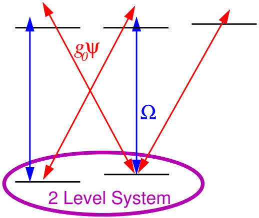
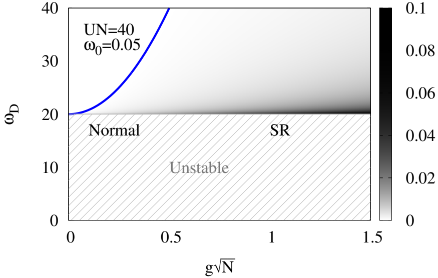
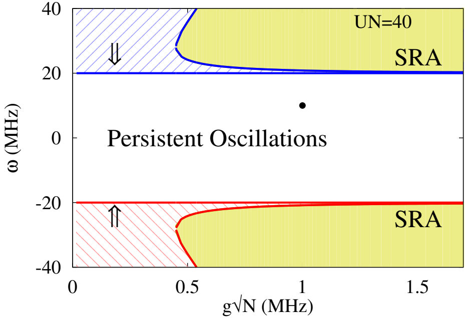
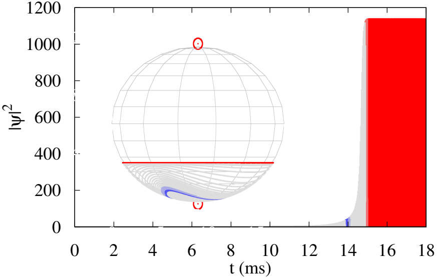




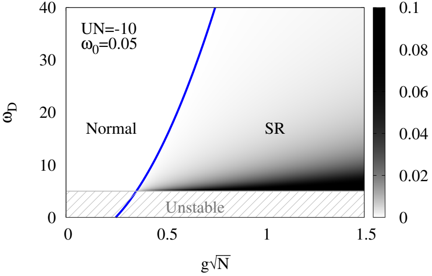


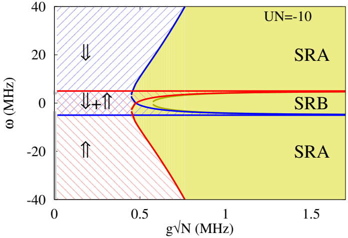


Outline

Open system and Beyond Dicke?

$UN = -10$MHz figures





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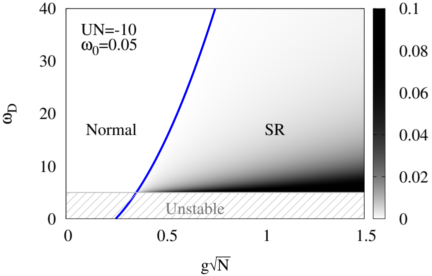
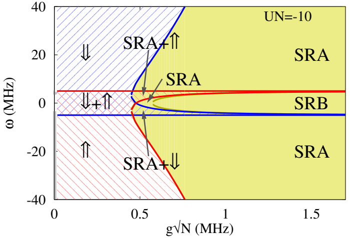
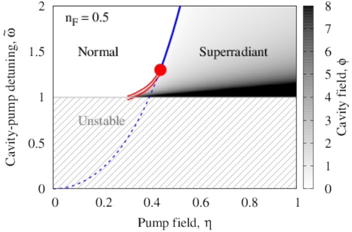




From fermions, found:



Survives to low n_F: Bosons!

- **But** needs state $\phi(x, z) = \cos(\sqrt{2}x)$
- **Missed by Dicke model**

Bosons beyond Dicke

BEC self organisation: $|\Psi_{\text{atoms}}\rangle = \left(\sum_{\mathbf{k}} \chi_{\mathbf{k}} a_{\mathbf{k}}\right)^N |0\rangle$, χ_k obeys:

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Truncate $|\mathbf{k}| < n_M$, $n_M = 1 \rightarrow$ Dicke

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- Hysteresis – Larkin-Pikin
- 2nd order at large $\tilde{\omega}$

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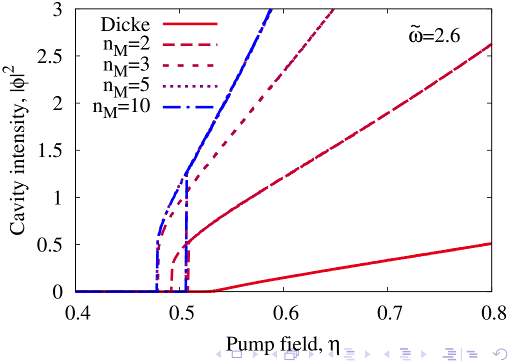

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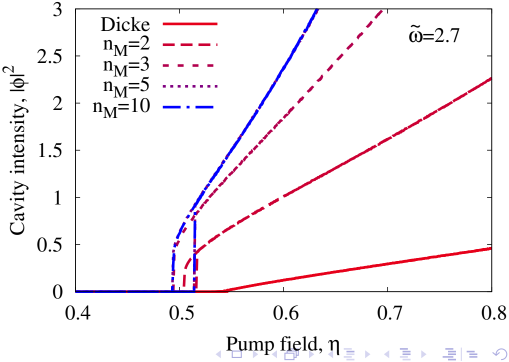

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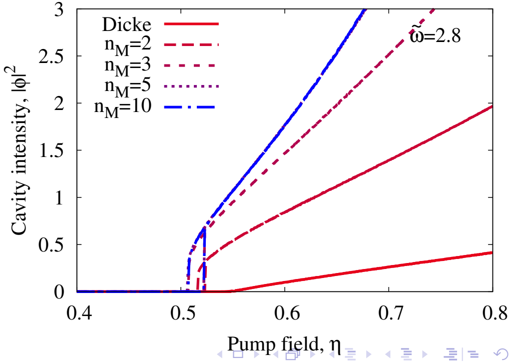

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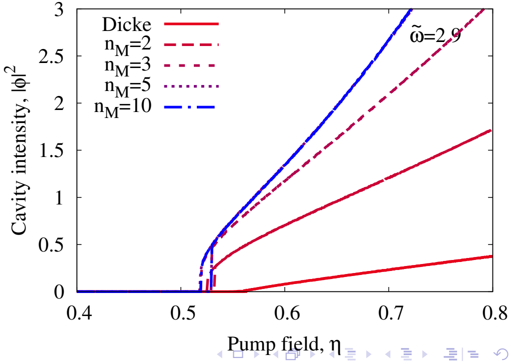

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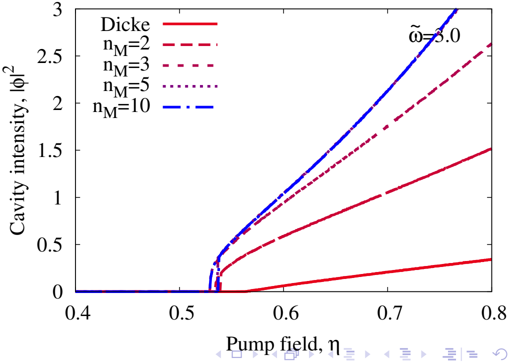
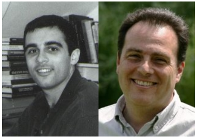


Acknowledgements

GROUP:



COLLABORATORS:



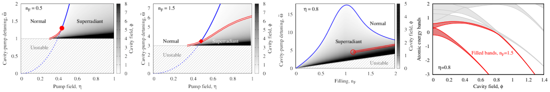

FUNDING:

topnes
Topological Protection and Non-Equilibrium States in Strongly Correlated Electron Systems

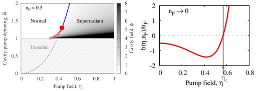
EPSRC
Engineering and Physical Sciences Research Council

Summary

- Fermions self organisation, liquid gas, and multicritical points

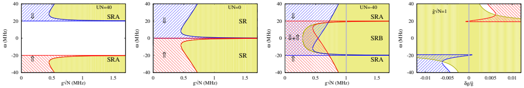


- First order transitions for bosons, outside Dicke model



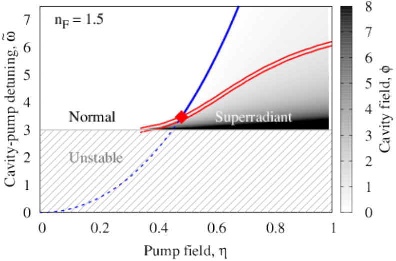
JK, Bhassen, Simons PRL '14

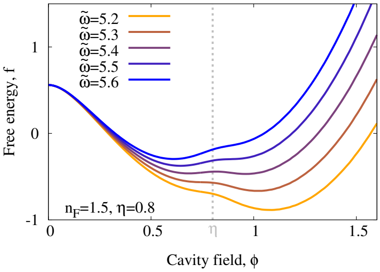
- Bosons: Dicke model shows many dynamical phases



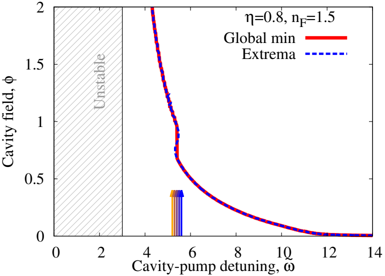
JK *et al.* PRL '10, Bhaseen *et al.* PRA '12

Bistability, signatures





Narrow bistable region



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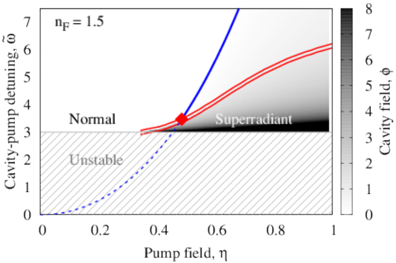
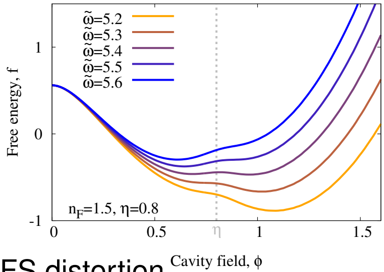
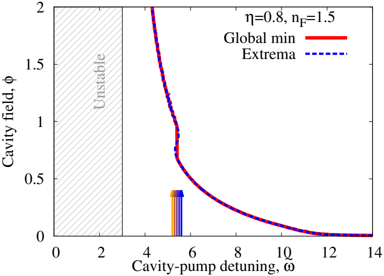
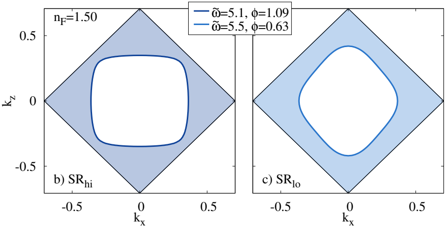




Narrow bistable region

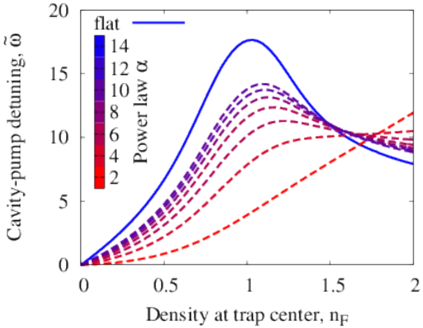
FS distortion





Fermi gas in a trap

- Trapped gas, $V(r) = E_R(r/r_0)^\alpha$
- Rescale via $\mathcal{A} = \pi r_0^2$
- Commensuration visible if flat enough ($\alpha > 4$)



Classical dynamics of the extended Dicke model

Open dynamical system:

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + U S_z \psi^\dagger\psi.$$

$$\partial_t \rho = -i[H, \rho] - \kappa(\psi^\dagger\psi\rho - 2\psi\rho\psi^\dagger + \rho\psi^\dagger\psi)$$

- Neglects quantum fluctuations
- Linearisation about fixed point → stability, spectrum

[JK *et al.* PRL '10, Bhaseen *et al.* PRA '12]

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Classical EOM
($|\mathbf{S}| = N/2 \gg 1$)

$$\dot{S}^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^*)S^z$$

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- $\psi = 0, \mathbf{S} = (0, 0, \pm N/2)$ always a solution.
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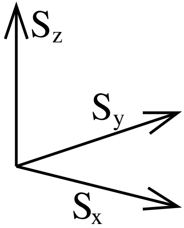

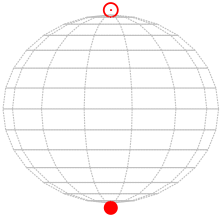

Small g: $\Uparrow, \Downarrow$ only.
($\omega = 30$MHz, $UN = -40$MHz)

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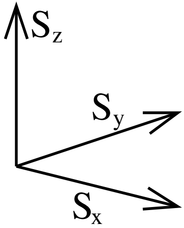
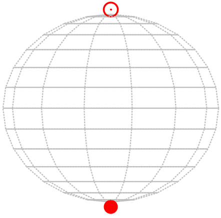
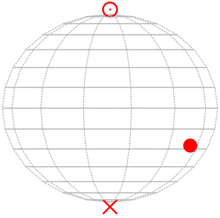


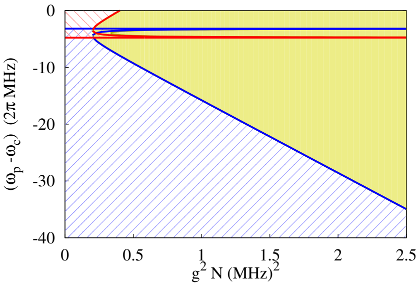


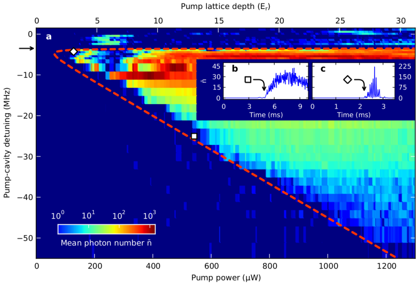


Small g: ⇑, ⇓ only. Larger g: SR too.
$(\omega = 30\text{MHz}, UN = -40\text{MHz})$

Outline

Comparison to experiment: $UN = -10\text{MHz}$





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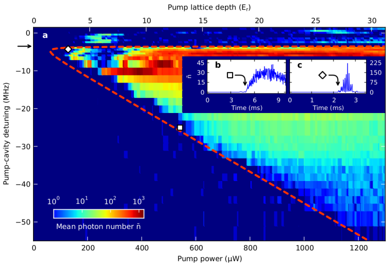
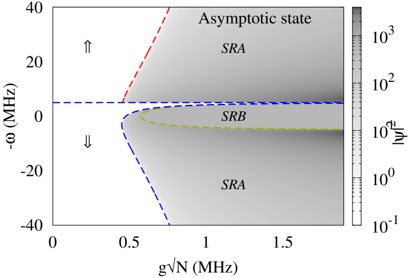
Adapted from: [Bhaseen *et al.* PRA '12]

[Baumann *et al* Nature '10]

Timescale to reach steady state





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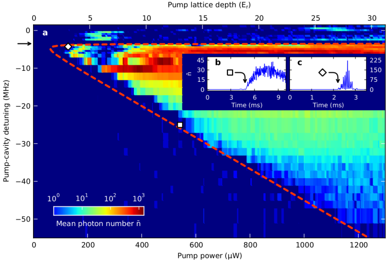

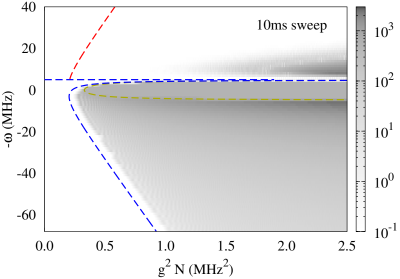

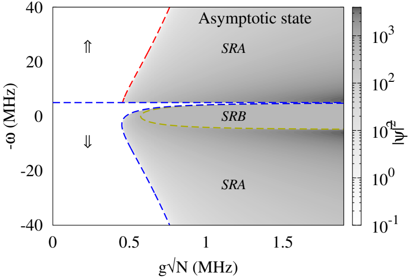

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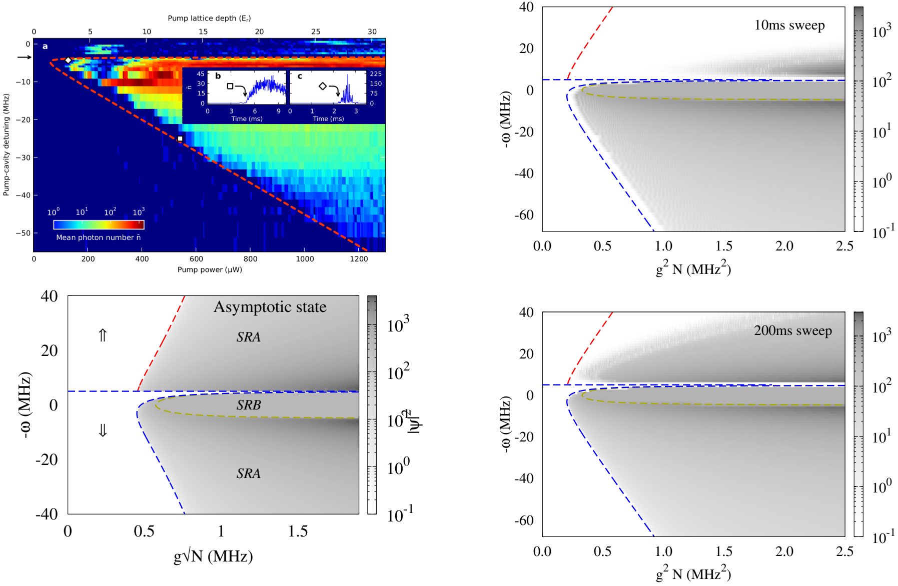
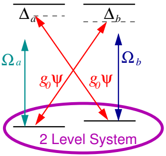


Timescales for dynamics: Why so slow and varied?

Suppose co- and counter-rotating terms differ



$$H = \ldots + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + \ldots$$

- SR(A) near phase boundary at small δg → Critical slowing down
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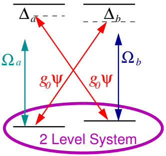

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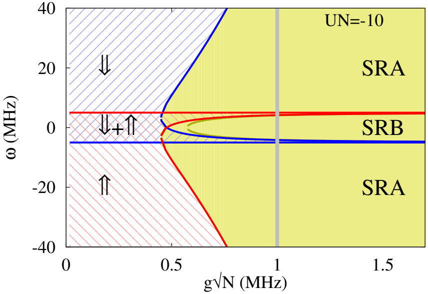

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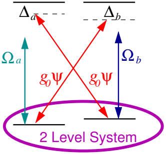

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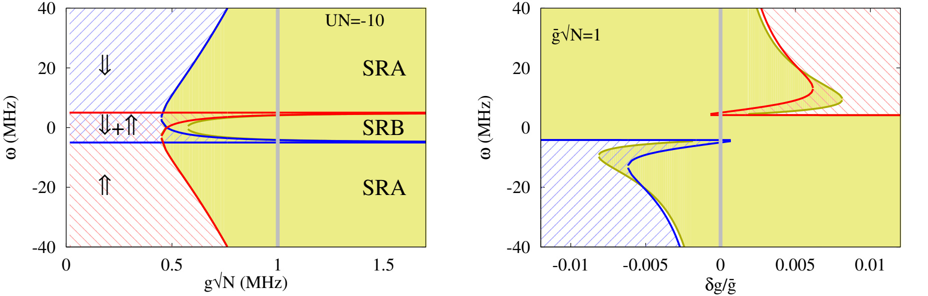

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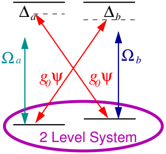

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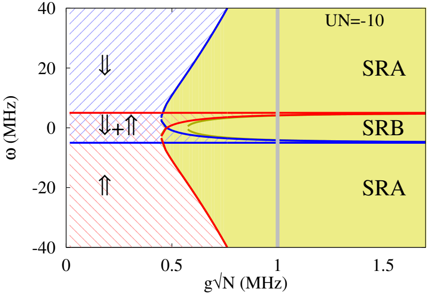

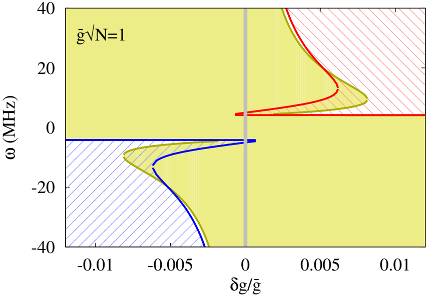

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Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

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(nb $g^2, \zeta, \eta \propto 1/V$).

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Gauge transform to dipole gauge $\mathbf{D} \cdot \mathbf{r}$

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + \bar{g}(S^+ - S^-)(\psi - \psi^\dagger)$$

“Dicke” transition at $\omega_0 < N\bar{g}^2/\omega \equiv 2\eta N$

But, ψ describes **electric displacement**

Grand canonical ensemble

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- If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)|\omega_0 - \mu|$

[Eastham and Littlewood, PRB ’01]

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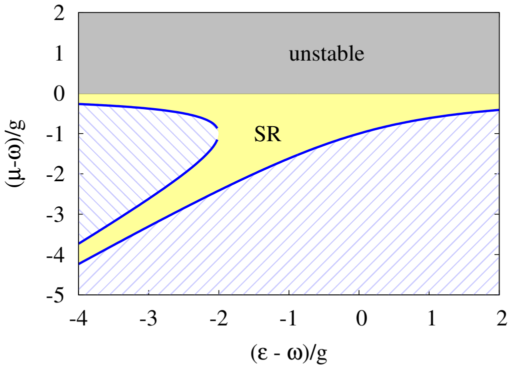

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