Superradiance of cold atoms in optical cavities

Jonathan Keeling

University of St Andrews
600 YEARS

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Coupling many atoms to light

Old question: *What happens to radiation when many atoms interact "collectively" with light.*

Superradiance — dynamical and steady state.

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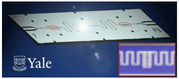
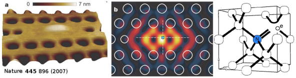
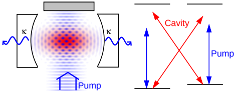
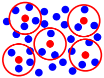
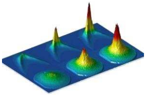
New relevance

- Superconducting qubits 

- Quantum dots & NV centres 

- Ultra-cold atoms 

- Rydberg atoms/polaritons 
- Microcavity Polaritons 

Dicke effect: Superradiance

PHYSICAL REVIEW VOLUME 93, NUMBER 1 JANUARY 1, 1954

Coherence in Spontaneous Radiation Processes

R. H. Dicke
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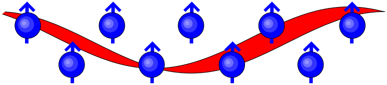

$$H_{\text{int}} = \sum_{k,i} g_k (\psi_k e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.})(S_i^+ + S_i^-)$$

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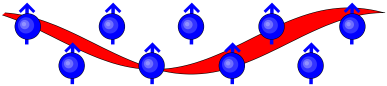

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If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$

Collective decay:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+S^-\rho - S^-\rho S^+ + \rho S^+ S^-\right]$$

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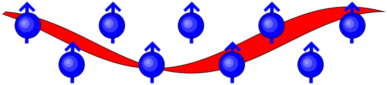

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If $S^z = |S| = N/2$ initially:

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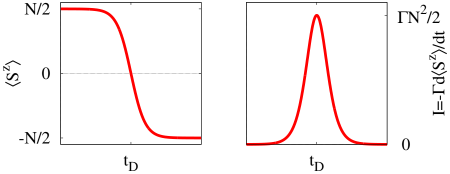

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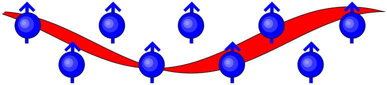

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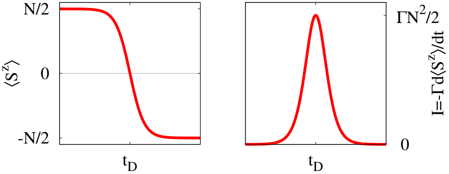
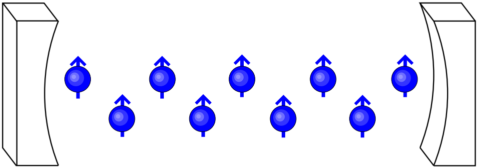


Problem: dipole interactions dephase. [Friedberg et al, Phys. Lett. 1972]

Collective emission with a cavity



One-mode: Oscillations

RWA $\rightarrow$ Tavis–Cummings model: $H_{\rm int} = \sum_i \left(\psi^\dagger S_i^- + \psi S_i^+ \right)$

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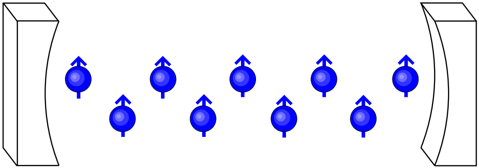

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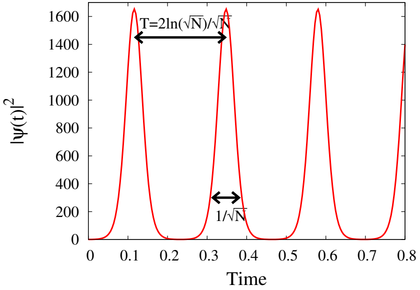

[Bonifacio and Preparata PRA '70]

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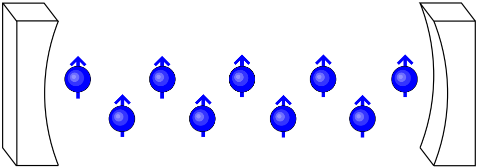

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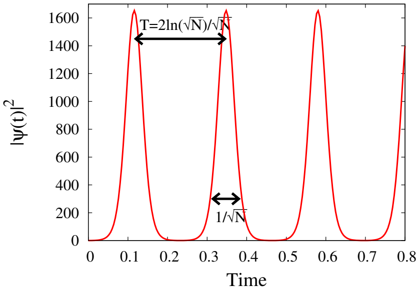

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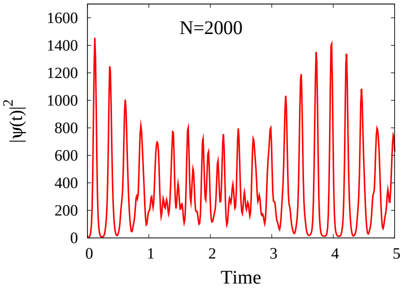
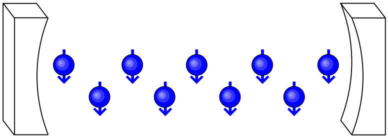


[JK PRA '09]

Dicke model and Dicke-Hepp-Lieb transition



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- Coherent state: $|\Psi\rangle \to e^{\lambda\psi^\dagger + \eta S^+}|\Omega\rangle$
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[Hepp, Lieb, Ann. Phys. '73]

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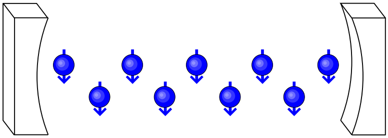

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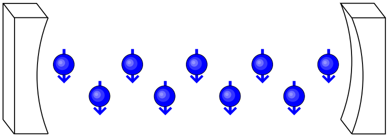

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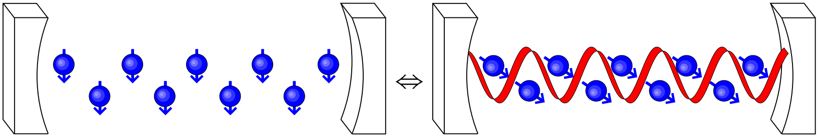

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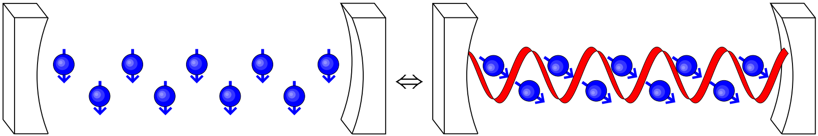

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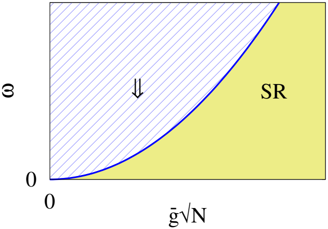

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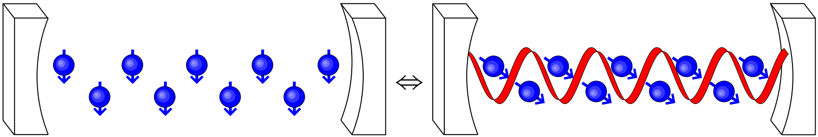
No go theorem for Dicke-Hepp-Lieb transition



Spontaneous polarisation if: $4Ng^2 > \omega\omega_0$

[Rzazewski *et al* PRL '75]

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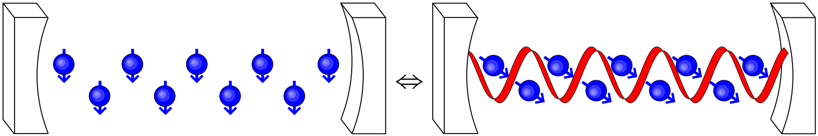

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No go theorem:. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \Leftrightarrow g(\psi^\dagger + \psi)(S^- + S^+), \qquad \sum_i \frac{A^2}{2m} \Leftrightarrow N\zeta(\psi + \psi^\dagger)^2$$

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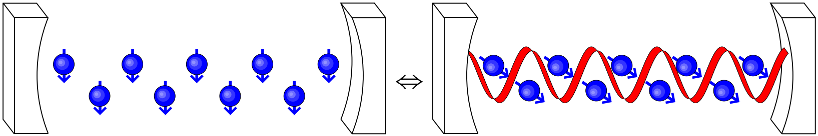

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For large N, $\omega \rightarrow \omega + 4N\zeta$. (RWA)

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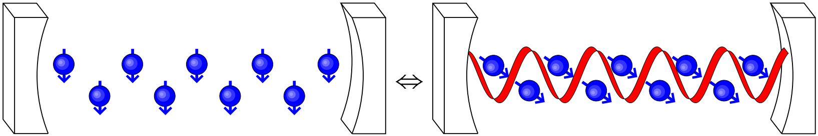

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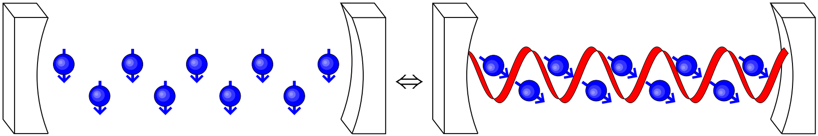

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But f-sum rule states: $g^2/\omega_0 < \zeta$. **No transition**

[Rzazewski *et al* PRL '75]

Ways around the no-go theorem

Problem: $g^2/\omega_0 < \zeta$ for intrinsic parameters. **Solutions**:

1. Gauge/interpretation of "photon"
   Ferroelectric transition in $\mathbf{D}\cdot\mathbf{r}$ gauge.
   [JK JPCM '07, Vukics & Domokos PRA 2012]
   - Circuit QED [Nataf and Ciuti, Nat. Comm. '10; Viehmann *et al.* PRL '11]
2. Grand canonical ensemble:
   - If $H \rightarrow H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
   - Incoherent pumping — polariton condensation.
3. Dissociate g, ω_0.
   e.g. Raman scheme: $\omega_0 \ll \omega$.
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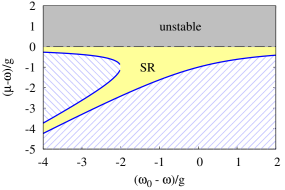

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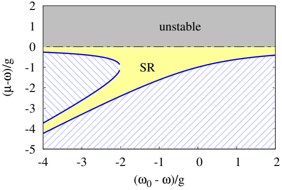

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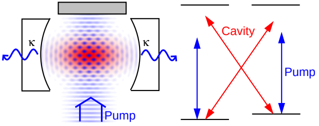

Outline

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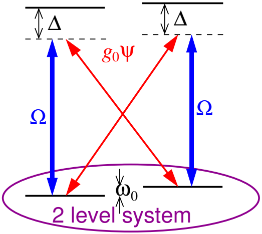

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- Imbalanced case (internal states): $H = \omega_0 S^z + g(\psi S^+ + \psi^\dagger S^-) + g'(\psi S^- + \psi^\dagger S^+) + \omega \psi^\dagger \psi$

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[Dimer *et al.* PRA '07]

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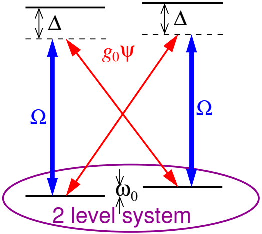

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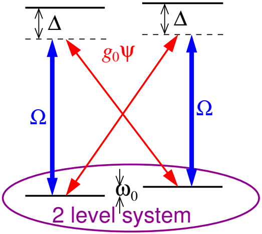

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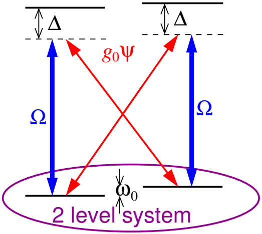
[Dimer *et al.* PRA '07]

Raman scheme, decoupling g, ω_0



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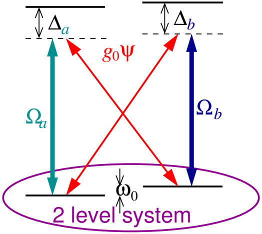

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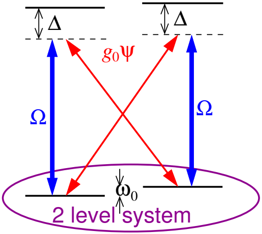

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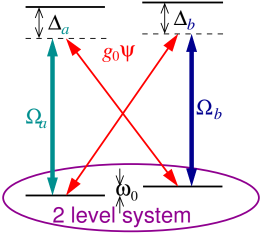
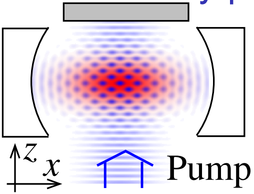


[Dimer *et al.* PRA '07]

Transverseley pumped cavity



Internal state → momentum states

1. Full description

$$H_0 = \omega_{\text{cavity}}\psi^\dagger\psi + \int d^2r \Big[\sum_{\alpha=e,g} c^\dagger_\alpha \left(\frac{-\nabla^2}{2m}\right) c_\alpha + \omega_{\text{atom}} c^\dagger_e c_e + E(x,z,t)(c^\dagger_e c_g + c^\dagger_g c_e) \Big]$$

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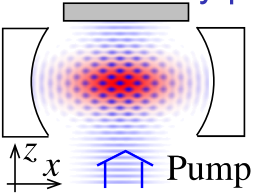

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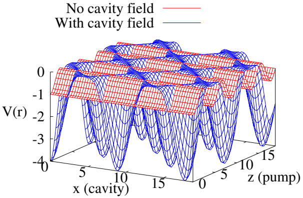

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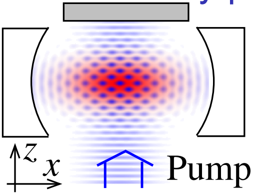

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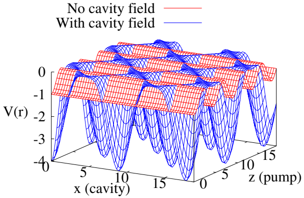

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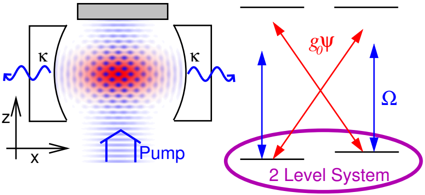
3. Dicke: project to atomic states $\phi(x,z) \propto \begin{cases} 1 \\ \cos(qz)\cos(qz) \end{cases}$

Mapping transverse pumping to Dicke model



Reduced basis:

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[Baumann *et al* Nature '10]

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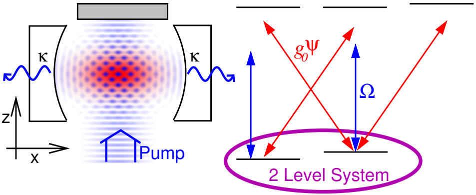

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"Feedback" due to extra states $U = -\frac{g_0^2}{4\Delta}$

[Baumann *et al* Nature '10]

Phase diagram of extended Dicke model

Ground state energy, $\lambda = \langle\psi\rangle/\sqrt{N}$:

$$\frac{E}{N} = \omega\lambda^2 - \frac{N}{2}\sqrt{(\omega_0 + UN\lambda^2)^2 + 16g^2N\lambda^2}$$

- Superradiant transition:

$$4g^2N > \left(\omega - \frac{UN}{2}\right)\omega_0$$

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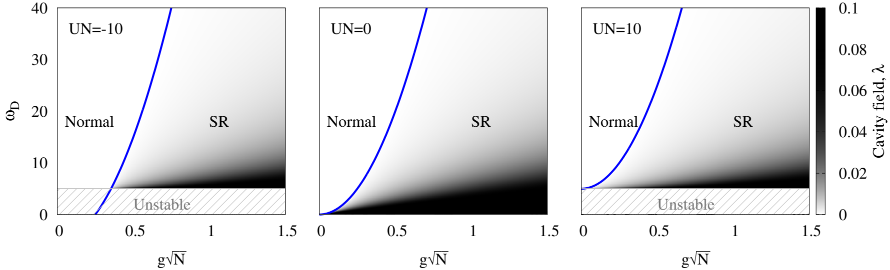
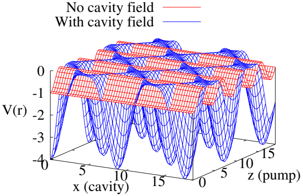


Outline

1. Dicke model, superradiance and no–go theorem
2. Superradiance and self-organisation
   - Raman scheme
   - Rayleigh scheme and hierarchies of H_{eff}
   - Generalized Dicke equilibrium theory
3. Fermionic self organisation
   - Equilibrium phase diagrams
   - Landau theory and microscopics
   - Evolution with filling
4. Open system dynamics
   - Linear stability with losses
   - Attractors of the Dicke model phases
   - Dicke model timescales
5. Conclusions

Fermions in optical cavities



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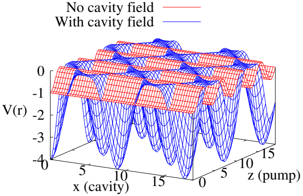

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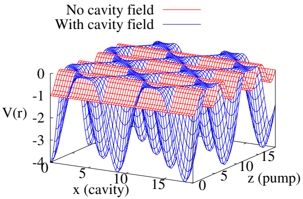

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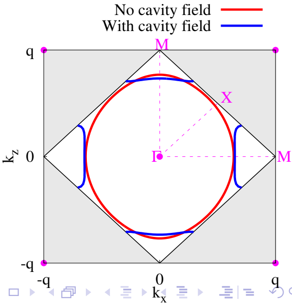

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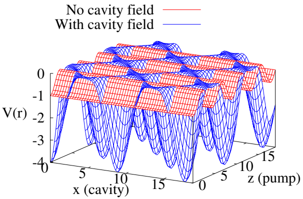

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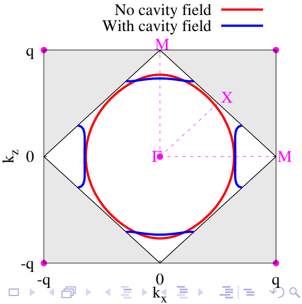
[JK, Bhaseen, & Simons; Piazza & Strack; Chen *et al.* All PRL '14.]



Dimensionless variables and free energy

- Rescale with $\sqrt{2}q, \omega_r = \hbar^2 q^2/2m$, Dimensionless variables:
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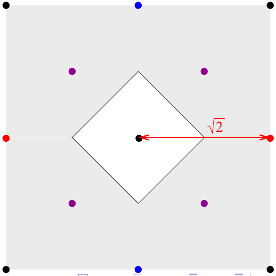

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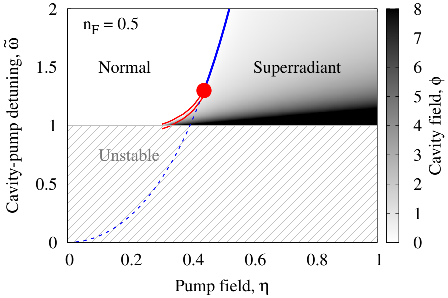

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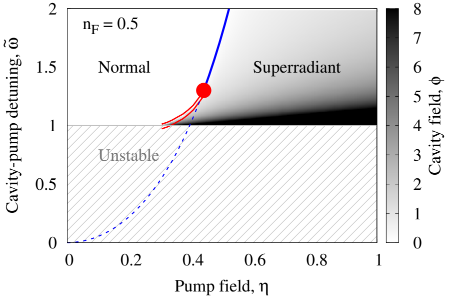

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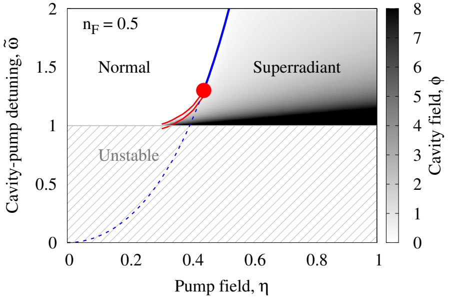

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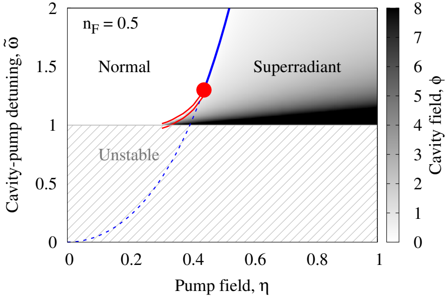
$$f = a\phi^2 + b\phi^4 + c\phi^6$$

$b < 0$ at small η.

Origin of first order transition



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Landau expansion: $f = a(\tilde{\omega}, \eta, n_F)\phi^2 + b(\eta, n_F)\phi^4 + c(\eta, n_F)\phi^6$

- Second order perturbation theory, $-\phi^4|m_{\mathbf{k},\mathbf{k}'}|^2/(E_{\mathbf{k}'} - E_{\mathbf{k}})$
- Larkin-Pikin like mechanism
- Survives to low n_F: Bosons!
  - But needs state $\phi(x,z) = \cos(\sqrt{2}x)$
  - Missed by Dicke model

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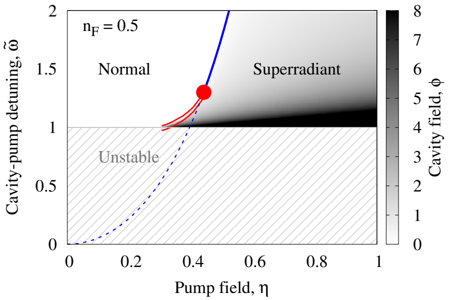

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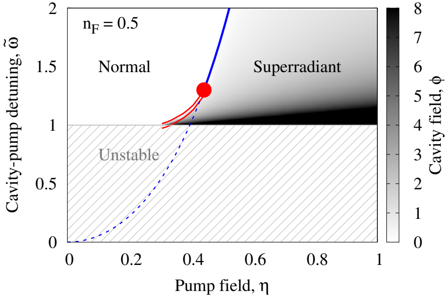
Landau expansion: $f = a(\tilde{\omega}, \eta, n_F)\phi^2 + b(\eta, n_F)\phi^4 + c(\eta, n_F)\phi^6$

- Second order perturbation theory,
  $-\phi^4 |m_{\mathbf{k},\mathbf{k}'}|^2 / (E_{\mathbf{k}'} - E_{\mathbf{k}})$
- Larkin-Pikin like mechanism
- Survives to low n_F: Bosons!
  - But needs state $\phi(x, z) = \cos(\sqrt{2}x)$
  - Missed by Dicke model

Origin of first order transition



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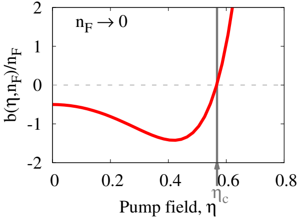

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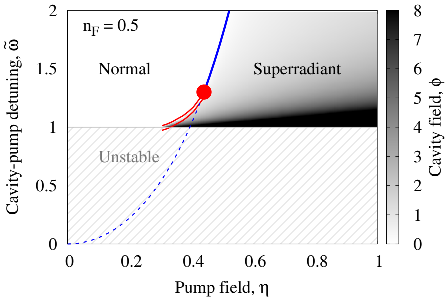

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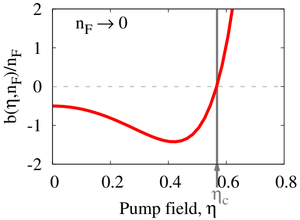
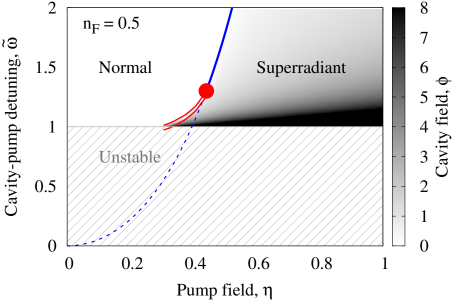


Higher fillings

$$f = a\phi^2 + b\phi^4 + c\phi^6$$

- Phase diagram unchanged for $n_F < 1$
- 2nd order line $a = 0$
- Tricritical ● at $a = b = 0$



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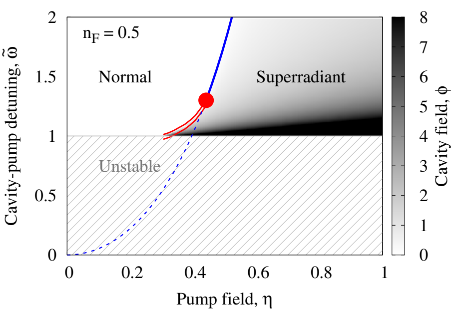

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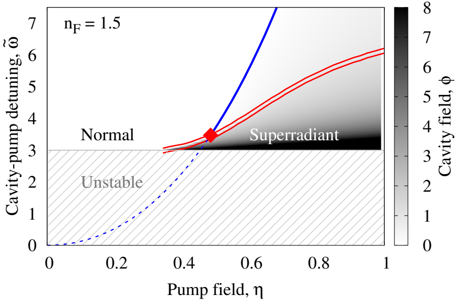

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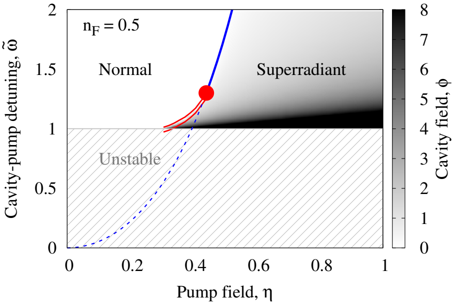

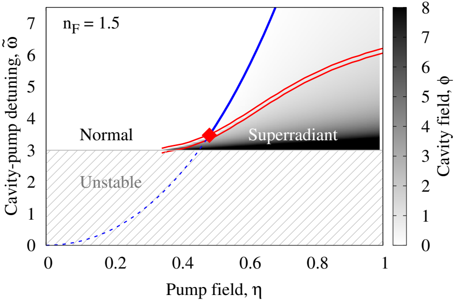

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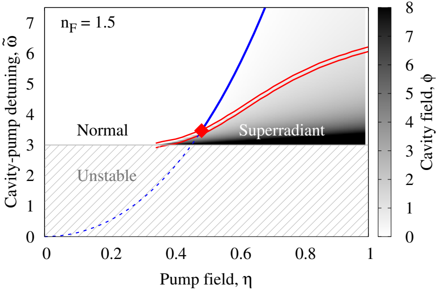

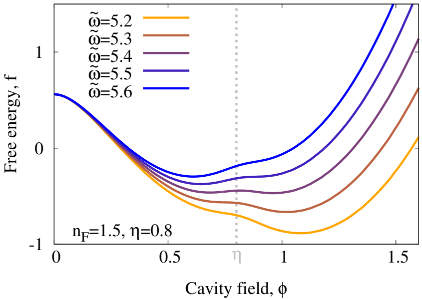

- $f(\phi) \to$ multiple minima
- Plot bands $\inf_k[\epsilon_{k,n}]$
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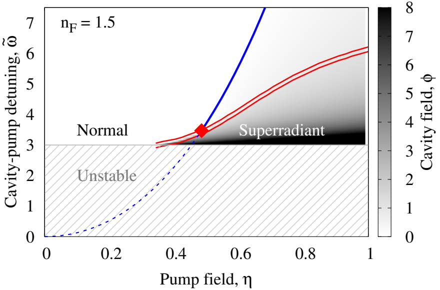

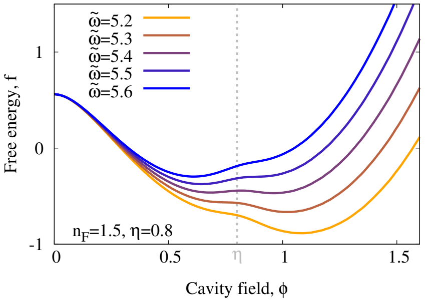

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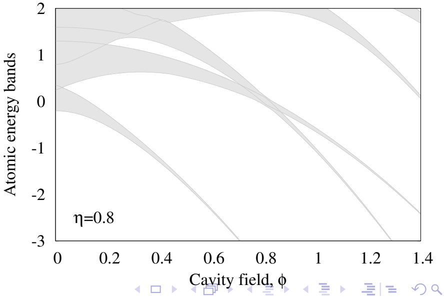

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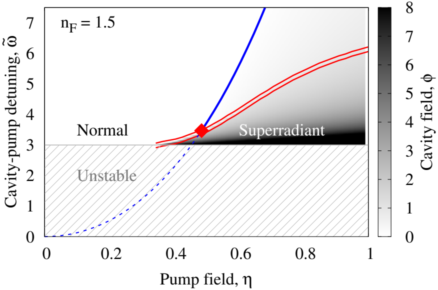

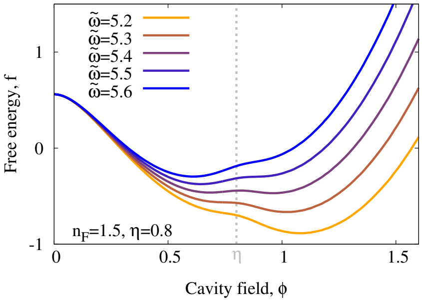

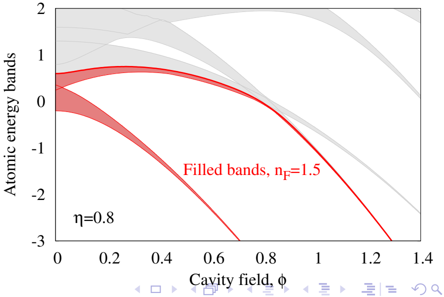

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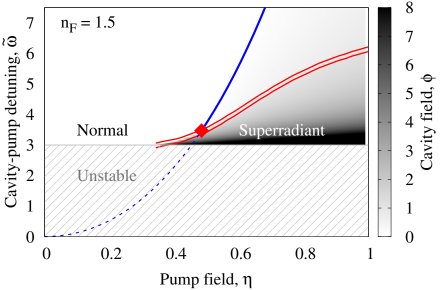

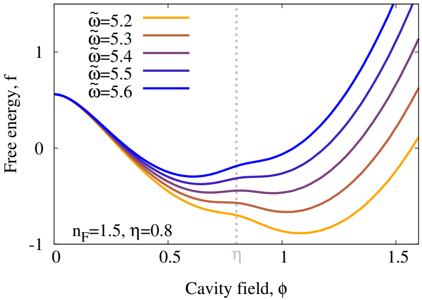

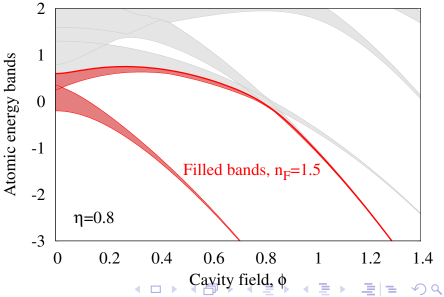

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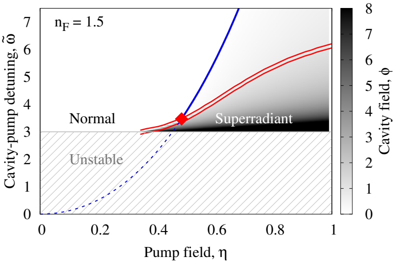
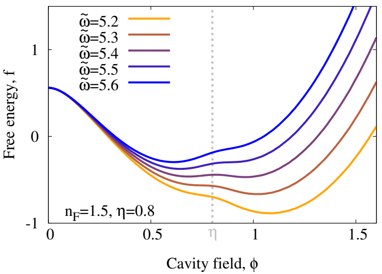
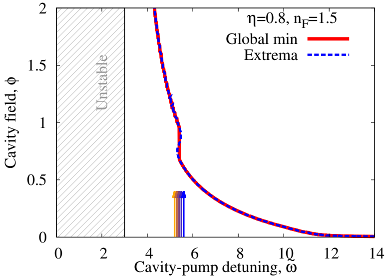
Bistability, signatures





Narrow bistable region



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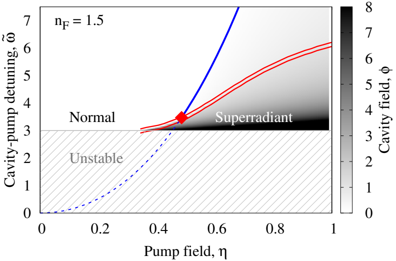

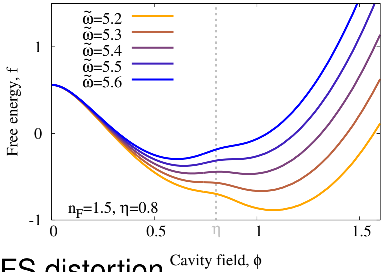

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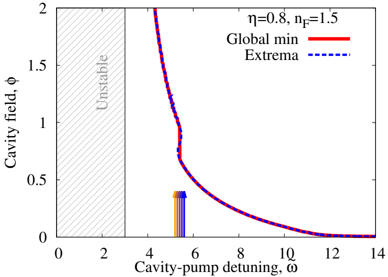
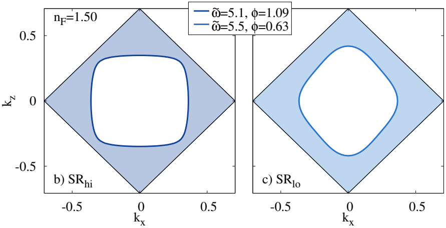
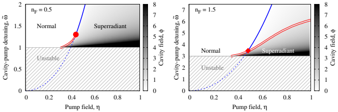
FS distortion





Phase diagram vs density

- Phase topology change:



- Fix η, plot vs n_F
- SR–SR after critical point ○
- Peak in 2nd order line $0 = a(\tilde{\omega}, n_F, \eta) = \tilde{\omega} + \chi(\eta, n_F)$
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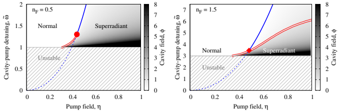

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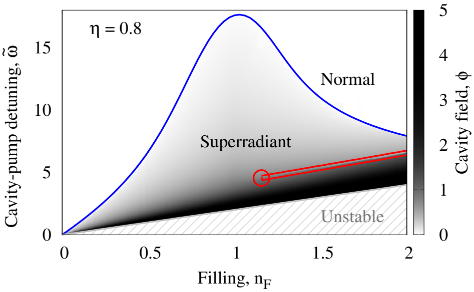

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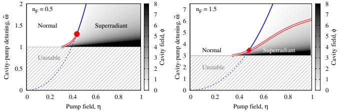

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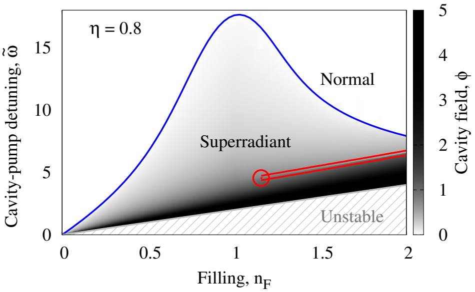

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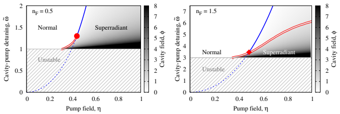

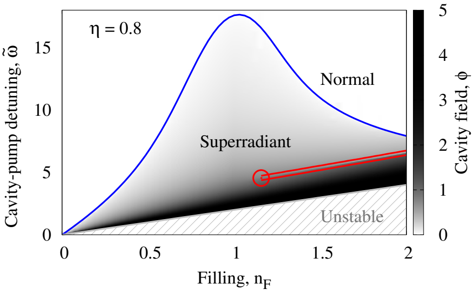

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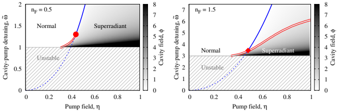

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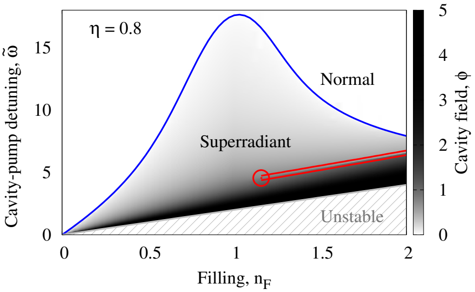

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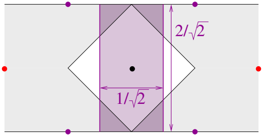

Outline

Open system vs ground state phase diagram

- Open system, $\dot{\rho} = -i[H, \rho] - \kappa\mathcal{L}[\psi]$. Stable attractors
- What survives — Normal-SR boundary
  - Fluctuations $\delta\phi = ue^{-i\nu t} + v^* e^{i\nu^* t}$.
- What must change
  - Unstable region $\rightarrow$ new attractors
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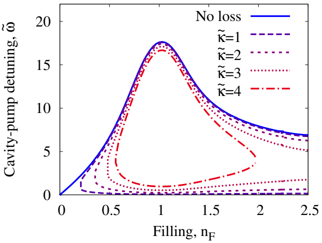

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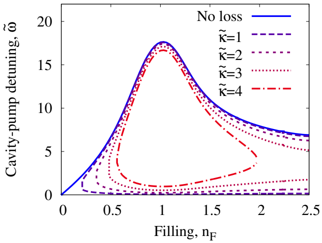

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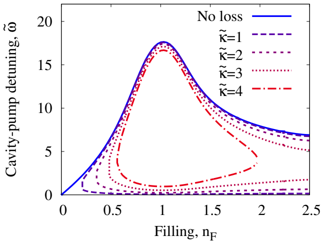

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Dicke model classical dynamics

Open dynamical system:

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + US_z\psi^\dagger\psi.$$

$$\partial_t\rho = -i[H,\rho] - \kappa(\psi^\dagger\psi\rho - 2\psi\rho\psi^\dagger + \rho\psi^\dagger\psi)$$

- Fixed points: $\dot{S} = 0, \dot{\psi} = 0$
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Classical EOM
($|\mathbf{S}| = N/2 \gg 1$)

$$\dot{S}^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^*)S^z$$

$$\dot{S}^z = ig(\psi + \psi^*)(S^- - S^+)$$

$$\dot{\psi} = -[\kappa + i(\omega + US^z)]\psi - ig(S^- + S^+)$$

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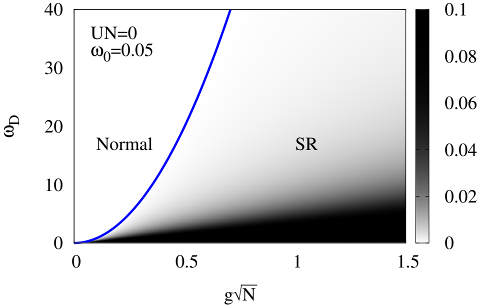
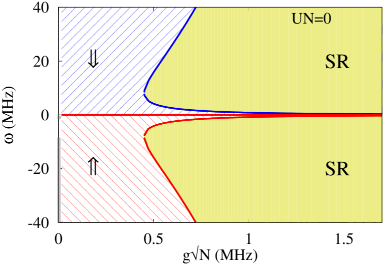
Long-time behaviour:

- Fixed points: $\dot{\mathbf{S}} = 0, \dot{\psi} = 0$
- Limit cycles?

Equilibrium Dicke vs open phase diagram, $UN = 0$





- Shift boundary $(\kappa^2 + \omega^2)/\omega = -\chi(\omega_0)$
- Allow negative ω → inverted

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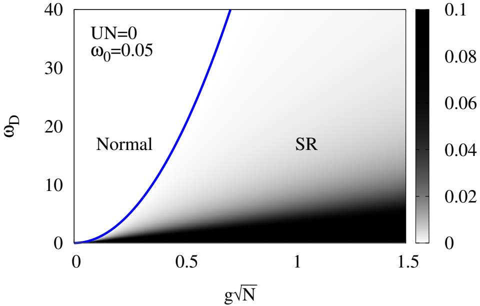

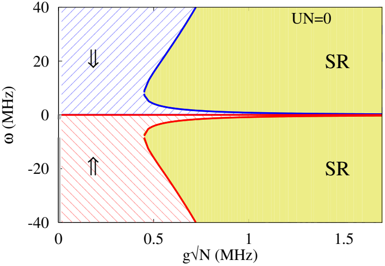

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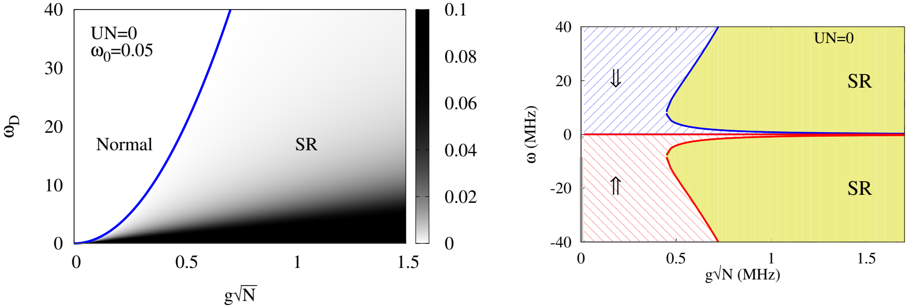

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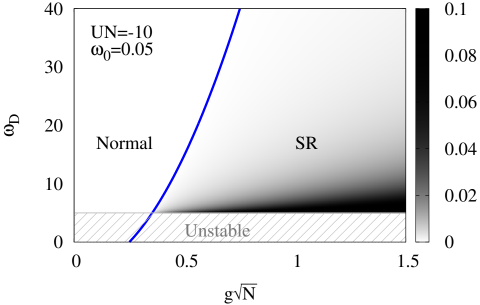
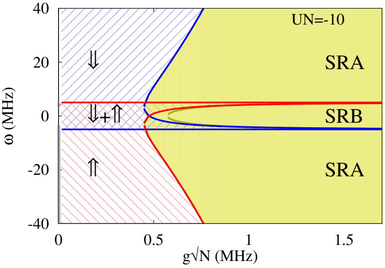
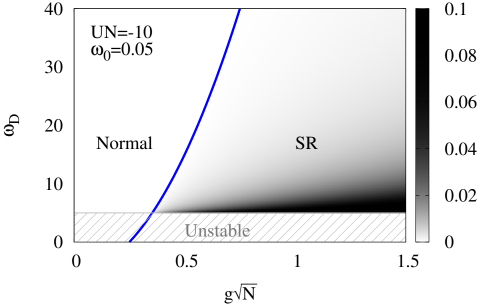
... Dicke ... $UN = -10\text{MHz}$

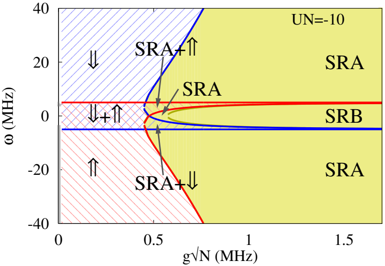




- Coexistence regions
- Unstable → SRB

. . . Dicke . . . $UN = -10$MHz





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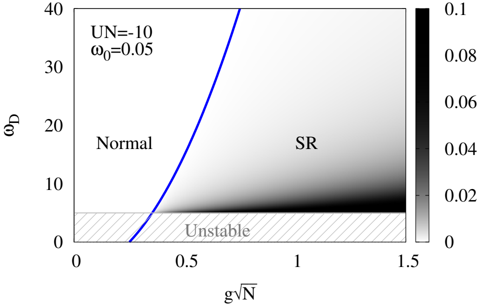

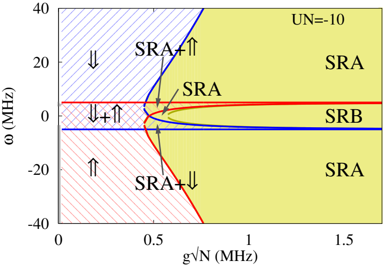

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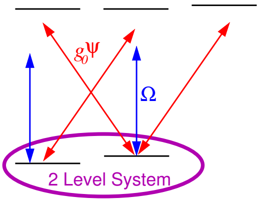
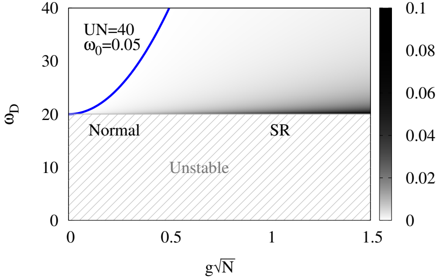
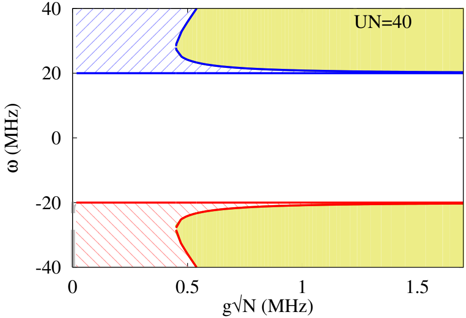
Changing U:







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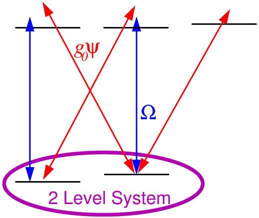

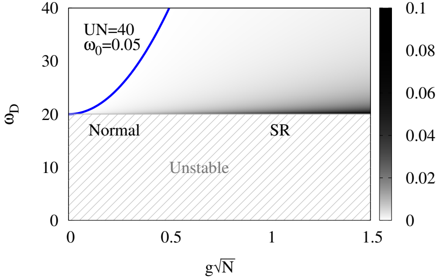

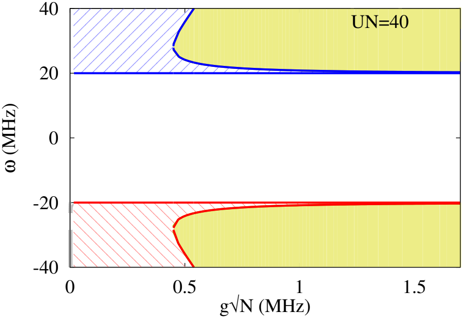

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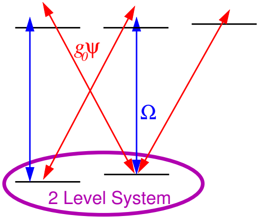
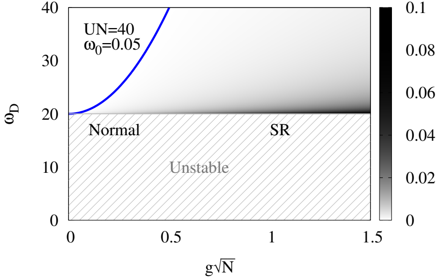
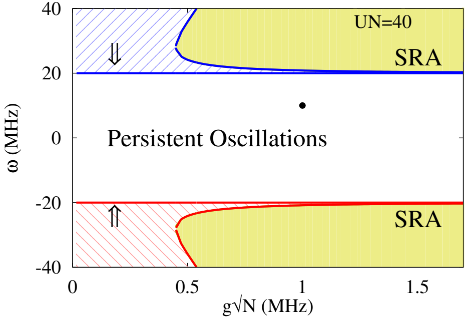
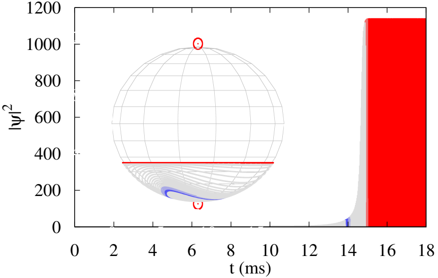
Changing *U*:



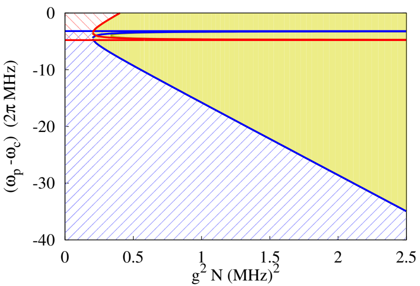


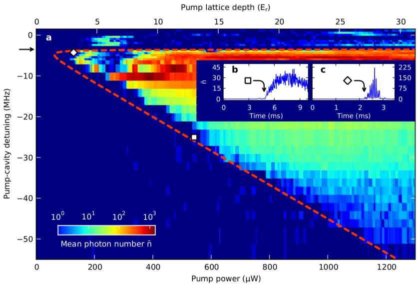




Outline

Comparison to experiment: $UN = -10\text{MHz}$



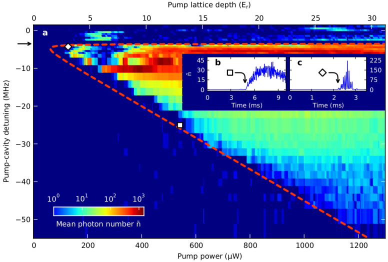


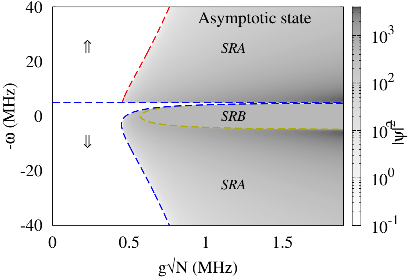
$UN = -10\text{MHz}$

Adapted from: [Bhaseen *et al.* PRA '12]

[Baumann *et al* Nature '10]

Timescale to reach steady state





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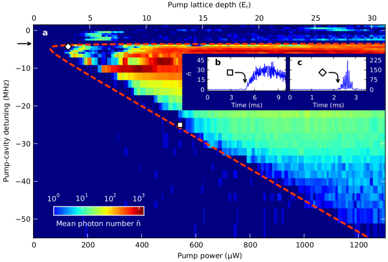

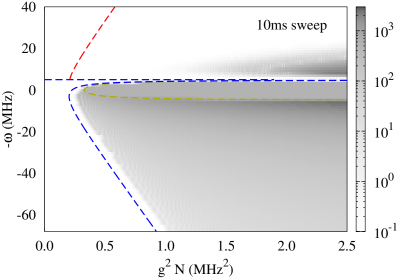

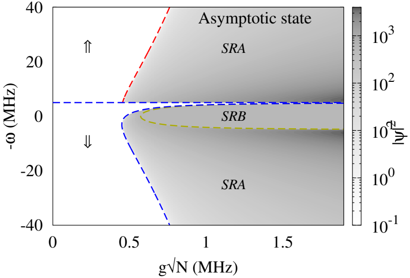

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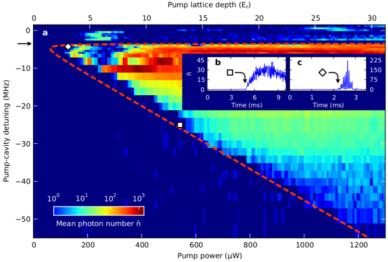
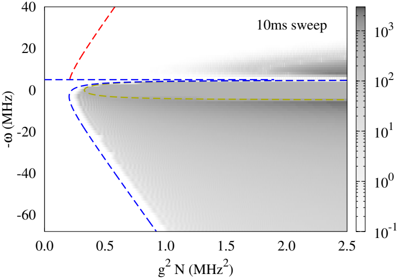
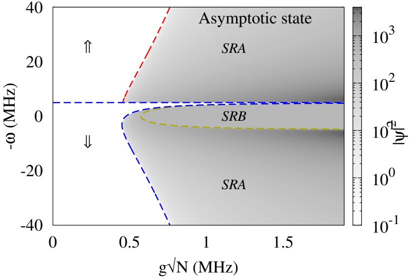
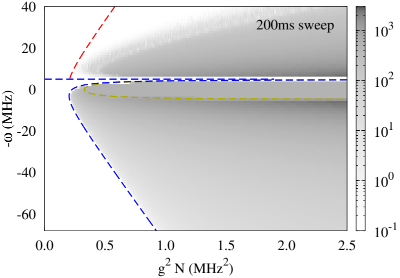


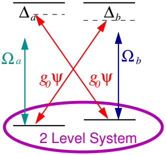






Timescales for dynamics: Why so slow and varied?

Suppose co- and counter-rotating terms differ



$$H = \ldots + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + \ldots$$

- SR(A) near phase boundary at small δg → Critical slowing down
- SR(A), SR(B) continuously connect

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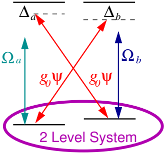

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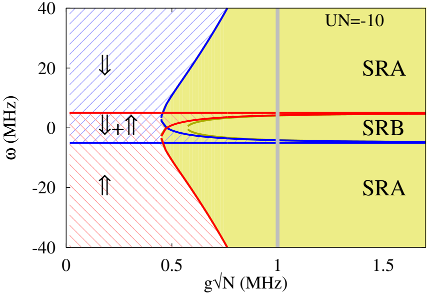

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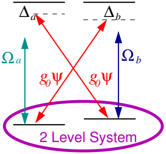

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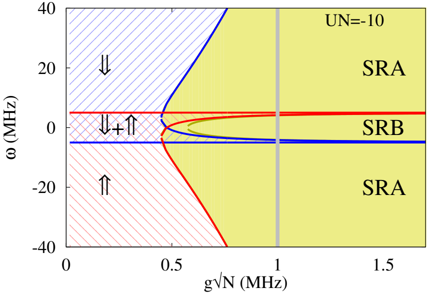

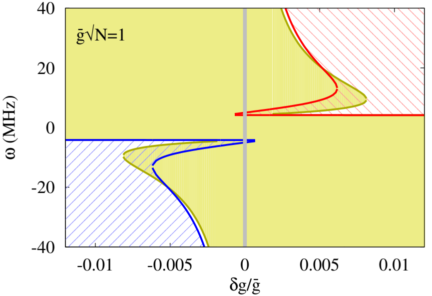

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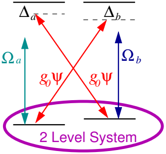

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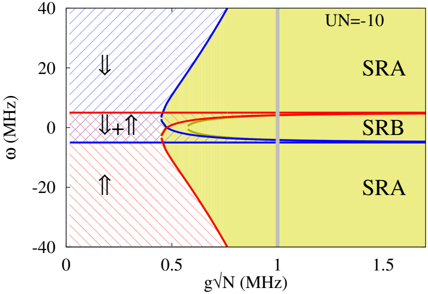

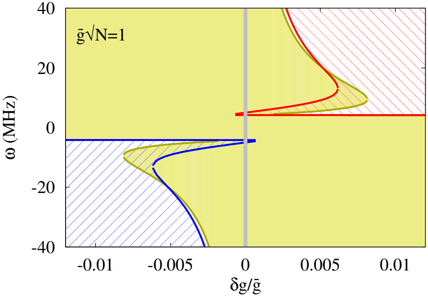

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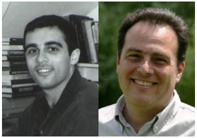
Acknowledgements

GROUP:

COLLABORATORS:



FUNDING:

topnes

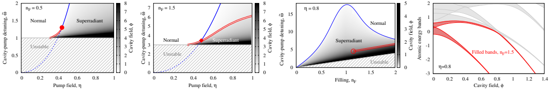
Topological Protection and Non-Equilibrium States in Strongly Correlated Electron Systems

EPSRC

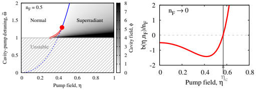
Engineering and Physical Sciences Research Council

Summary

- Fermions self organisation, liquid gas, and multicritical points

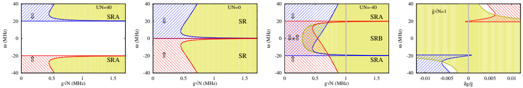


- First order transitions for bosons, outside Dicke model



JK, Bhassen, Simons PRL '14

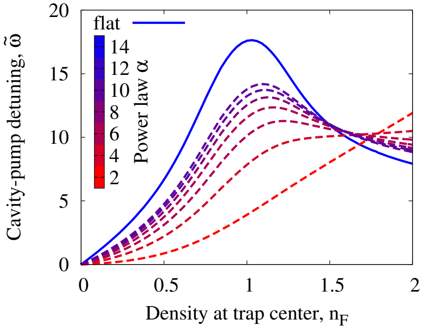
- Bosons: Dicke model shows many dynamical phases



JK *et al.* PRL '10, Bhaseen *et al.* PRA '12

Fermi gas in a trap

- Trapped gas, $V(r) = E_R(r/r_0)^\alpha$
- Rescale via $\mathcal{A} = \pi r_0^2$
- Commensuration visible if flat enough ($\alpha > 4$)



Classical dynamics of the extended Dicke model

Open dynamical system:

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + U S_z \psi^\dagger\psi.$$

$$\partial_t\rho = -i[H, \rho] - \kappa(\psi^\dagger\psi\rho - 2\psi\rho\psi^\dagger + \rho\psi^\dagger\psi)$$

- Neglects quantum fluctuations
- Linearisation about fixed point $\rightarrow$ stability, spectrum

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Fixed points (steady states)

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- $\psi = 0, \mathbf{S} = (0, 0, \pm N/2)$ always a solution.
- If $g > g_c$, $\psi \neq 0$ too
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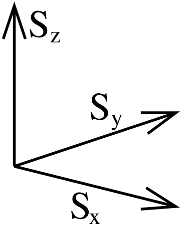

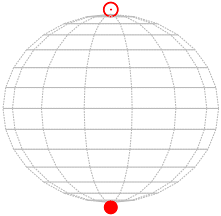

Small g: ⇑, ⇓ only.
($\omega = 30$MHz, $UN = -40$MHz)

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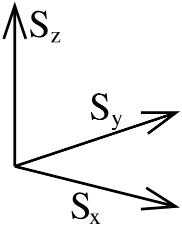

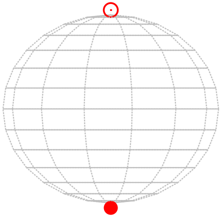

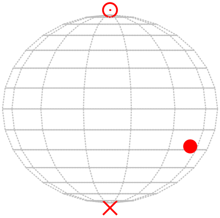

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Ferroelectric transition

Atoms in Coulomb gauge

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Two-level systems — dipole-dipole coupling

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + g(S^+ + S^-)(\psi + \psi^\dagger) + N\zeta(\psi + \psi^\dagger)^2 - \eta(S^+ - S^-)^2$$

(nb $g^2, \zeta, \eta \propto 1/V$).

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Ferroelectric polarisation if $\omega_0 < 2\eta N$

Gauge transform to dipole gauge $\mathbf{D} \cdot \mathbf{r}$

$$H = \omega_0 S^z + \omega \psi^\dagger \psi + \bar{g}(S^+ - S^-)(\psi - \psi^\dagger)$$

"Dicke" transition at $\omega_0 < N\bar{g}^2/\omega \equiv 2\eta N$

But, ψ describes **electric displacement**

Grand canonical ensemble

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- Fix density / fix $\mu > 0$ — pumping

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[Eastham and Littlewood, PRB '01]

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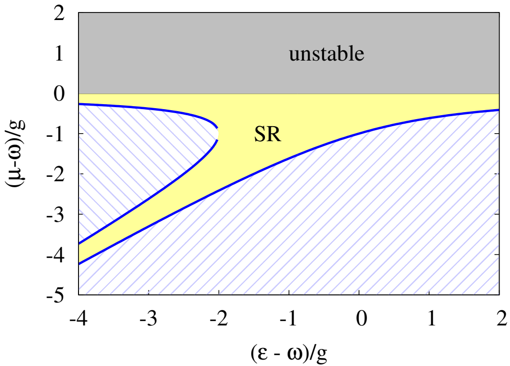

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