Superradiance of cold atoms in optical cavities

Jonathan Keeling

University of St Andrews
600 YEARS

Durham, October 2013

Coupling many atoms to light

Old question: *What happens to radiation when many atoms interact "collectively" with light.*

Superradiance — dynamical and steady state.

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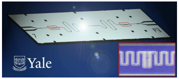
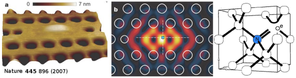
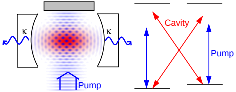
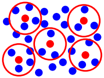
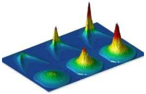
New relevance

- Superconducting qubits 

- Quantum dots & NV centres 

- Ultra-cold atoms 

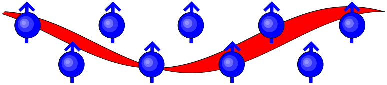
- Rydberg atoms/polaritons 
- Microcavity Polaritons 

Dicke effect: Superradiance

PHYSICAL REVIEW VOLUME 93, NUMBER 1 JANUARY 1, 1954

Coherence in Spontaneous Radiation Processes

R. H. Dicke
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey



$$H_{\text{int}} = \sum_{k,i} g_k (\psi_k e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.})(S_i^+ + S_i^-)$$

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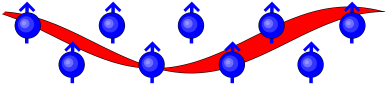

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If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$
Collective decay:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+S^-\rho - S^-\rho S^+ + \rho S^+ S^-\right]$$

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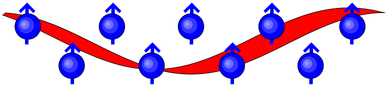

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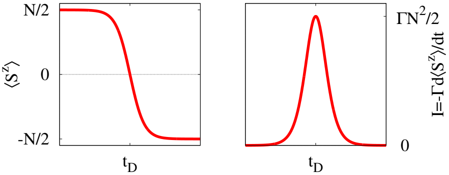

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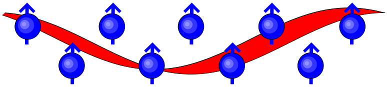

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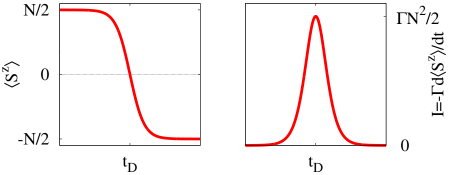
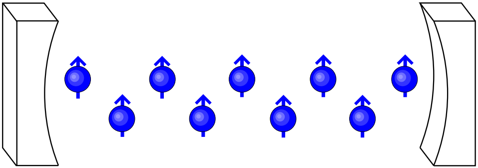


Problem: dipole interactions dephase. [Friedberg et al, Phys. Lett. 1972]

Collective emission with a cavity



One-mode: Oscillations

RWA → Tavis–Cummings model: $H_{\text{int}} = \sum_i \left(\psi^\dagger S_i^- + \psi S_i^+ \right)$

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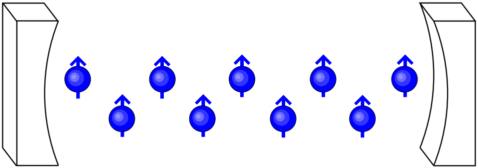

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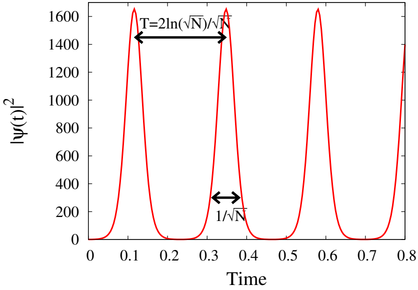

[Bonifacio and Preparata PRA '70]

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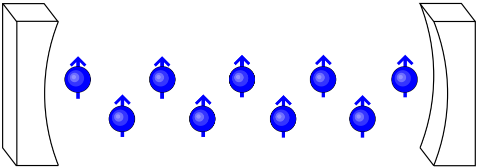

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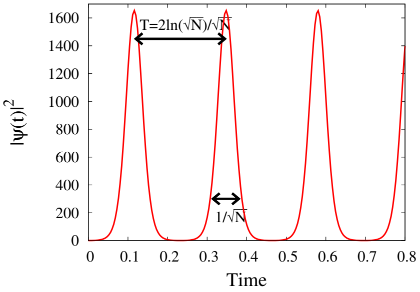

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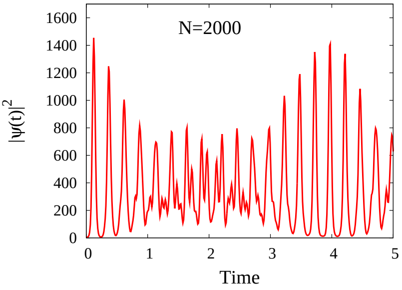
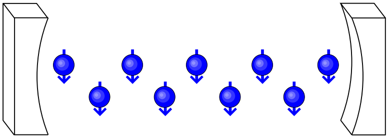


[JK PRA '09]

Dicke model and Dicke-Hepp-Lieb transition



$$H = \omega\psi^\dagger\psi + \sum_i \omega_0 S_i^z + g(\psi + \psi^\dagger)(S_i^+ + S_i^-)$$

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[Hepp, Lieb, Ann. Phys. '73]

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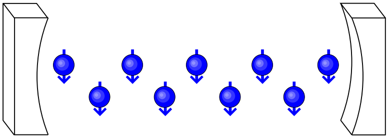

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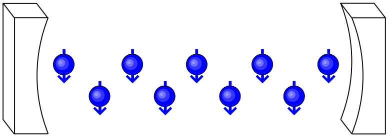

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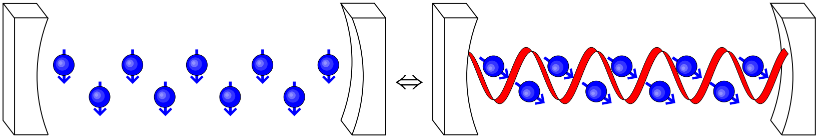

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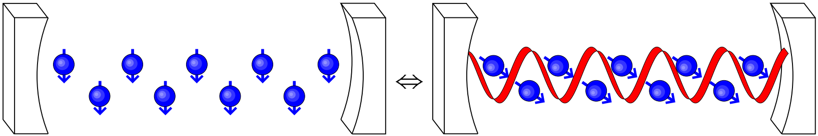

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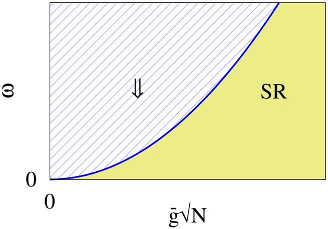

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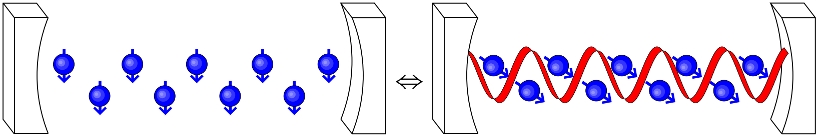
No go theorem for Dicke-Hepp-Lieb transition



Spontaneous polarisation if: $4Ng^2 > \omega\omega_0$

[Rzazewski *et al* PRL ’75]

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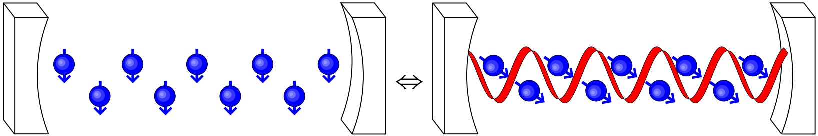

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No go theorem:. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \Leftrightarrow g(\psi^\dagger + \psi)(S^- + S^+), \qquad \sum_i \frac{A^2}{2m} \Leftrightarrow N\zeta(\psi + \psi^\dagger)^2$$

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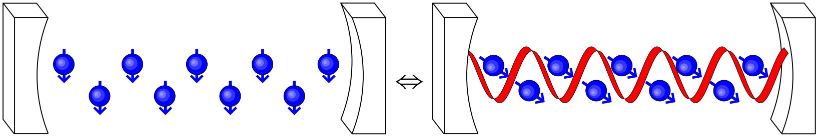

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For large N, $\omega \to \omega + 4N\zeta$. (RWA)

[Rzazewski *et al* PRL '75]

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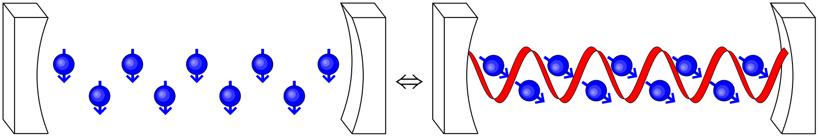

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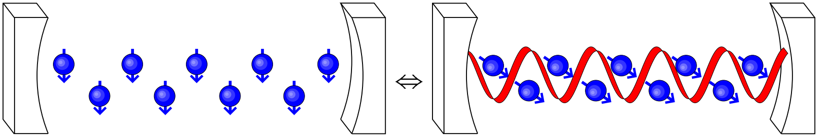

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But f-sum rule states: $g^2/\omega_0 < \zeta$. **No transition**

[Rzazewski *et al* PRL '75]

Ways around the no-go theorem

Problem: $g^2/\omega_0 < \zeta$ for intrinsic parameters. **Solutions**:

1. Gauge/interpretation of "photon"
   Ferroelectric transition in $\mathbf{D}\cdot\mathbf{r}$ gauge.
   [JK JPCM '07, Vukics & Domokos PRA 2012]
   - Circuit QED [Nataf and Ciuti, Nat. Comm. '10; Viehmann *et al.* PRL '11]
2. Grand canonical ensemble:
   - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2N > (\omega - \mu)(\omega_0 - \mu)$
   - Incoherent pumping — polariton condensation.
3. Dissociate g, ω_0.
   e.g. Raman scheme: $\omega_0 \ll \omega$.
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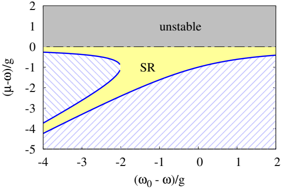

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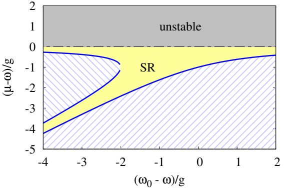

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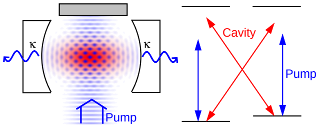

Outline

1. Dicke model, superradiance and no–go theorem
2. Superradiance and self-organisation
   - Raman scheme
   - Rayleigh scheme and hierarchies of H_{eff}
   - Generalized Dicke equilibrium theory
3. Fermionic self organisation
   - Equilibrium phase diagrams
   - Landau theory and microscopics
   - Evolution with filling
4. Open system dynamics
   - Linear stability with losses
   - Attractors of the Dicke model phases
   - Dicke model timescales
5. Conclusions

Acknowledgements

GROUP:

  

COLLABORATORS:

  

FUNDING:

EPSRC
Engineering and Physical Sciences
Research Council

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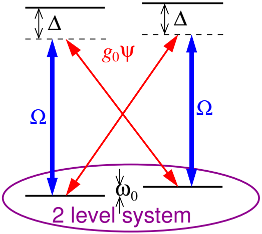

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- New "feedback" term $U = \frac{g_0^2}{2\Delta_b} - \frac{g_0^2}{2\Delta_a}$

[Dimer *et al.* PRA '07]

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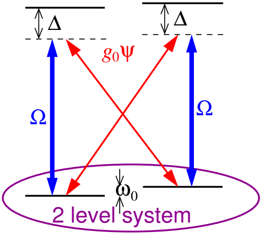

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[Dimer *et al.* PRA '07]

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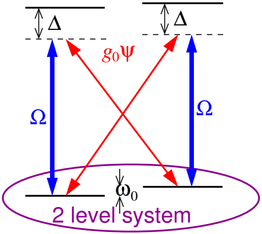

$$H = \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + \omega\psi^\dagger\psi$$

- 2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$
- Coupling $g = \dfrac{g_0\Omega}{2\Delta}$
- Rotating frame of pump, $\omega = \omega_{\text{cavity}} - \omega_{\text{pump}}$

- Imbalanced case (internal states): $H = \omega_0 S^z + g(\psi S^+ + \psi^\dagger S^-) + g'(\psi S^- + \psi^\dagger S^+) + \omega\psi^\dagger\psi$
- Imbalance: $g = \dfrac{g_0\Omega_b}{2\Delta_b} \neq g' = \dfrac{g_0\Omega_a}{2\Delta_a}$
- New “feedback” term $U = \dfrac{g_0^2}{2\Delta_b} - \dfrac{g_0^2}{2\Delta_a}$

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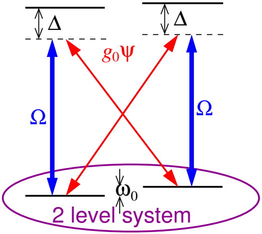

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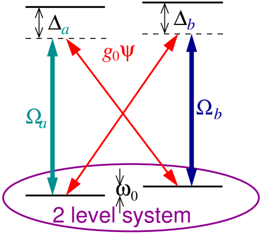

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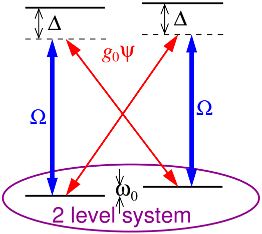

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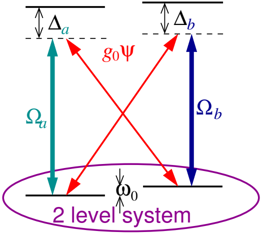
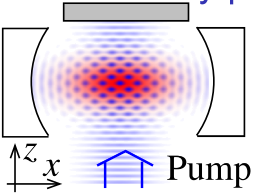


[Dimer *et al.* PRA '07]

Transversely pumped cavity



Internal state → momentum states

1. Full description

$$H_0 = \omega_{\text{cavity}} \psi^\dagger \psi + \int d^2 r \Big[\sum_{\alpha=e,g} c_\alpha^\dagger \left(\frac{-\nabla^2}{2m} \right) c_\alpha + \omega_{\text{atom}} c_e^\dagger c_e + E(x,z,t)(c_e^\dagger c_g + c_g^\dagger c_e) \Big]$$

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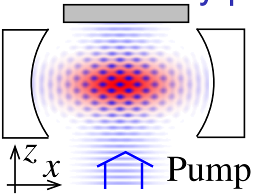

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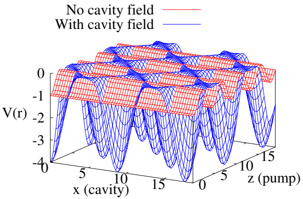

2. Eliminate e state
   - Rotating frame $\omega = \omega_{\text{cavity}} - \omega_{\text{pump}} - N\delta$

$$H = \omega \psi^\dagger \psi + \int d^2 \mathbf{r} c^\dagger(\mathbf{r}) \left(-\frac{\nabla^2}{2m} - V(\mathbf{r}) \right) c(\mathbf{r})$$

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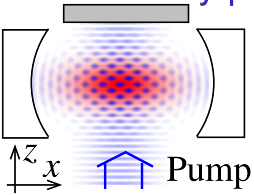

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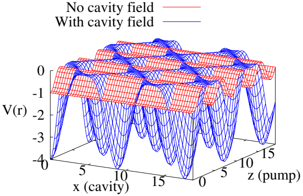

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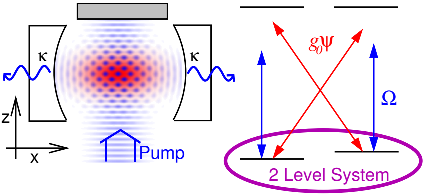
3. Dicke: project to atomic states $\phi(x,z) \propto \begin{cases} 1 \\ \cos(qz)\cos(qz) \end{cases}$

Mapping transverse pumping to Dicke model



Reduced basis:

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[Baumann *et al* Nature '10]

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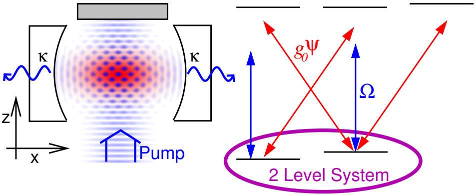

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"Feedback" due to extra states $U = -\frac{g_0^2}{4\Delta}$

[Baumann *et al* Nature '10]

Phase diagram of extended Dicke model

Ground state energy, $\lambda = \langle\psi\rangle/\sqrt{N}$:

$$\frac{E}{N} = \omega\lambda^2 - \frac{N}{2}\sqrt{(\omega_0 + UN\lambda^2)^2 + 16g^2N\lambda^2}$$

- Superradiant transition:

$$4g^2N > \left(\omega - \frac{UN}{2}\right)\omega_0$$

- Stability, $\lambda \to \infty$

$$E \sim \left(\omega - \frac{UN}{2}\right)\lambda^2 + \ldots$$

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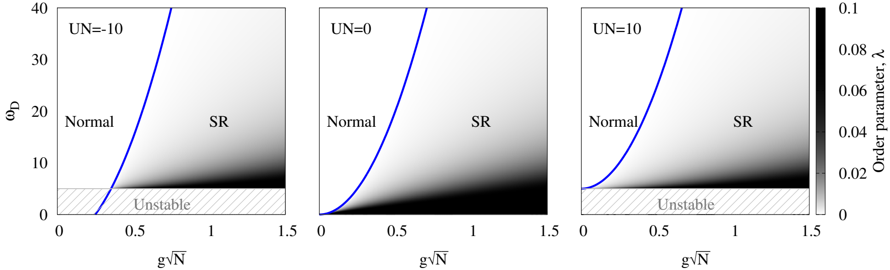
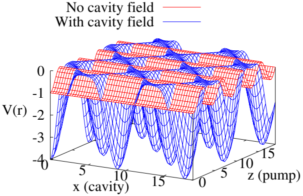


Outline

Fermions in optical cavities



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[Domokos & Ritsch, PRL '03; Black *et al.* PRL '03; Piazza, Strack, Zweger 1305.2928]

- Pauli blocking
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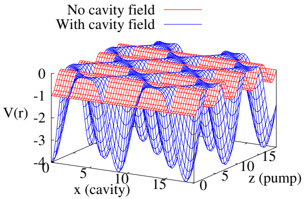

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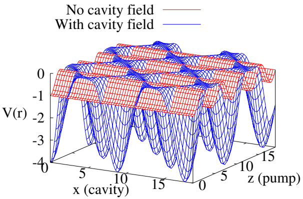

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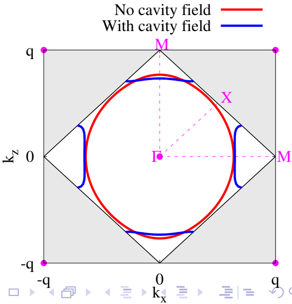

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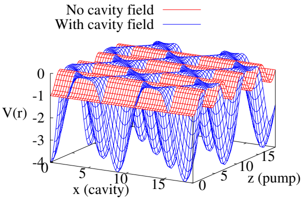

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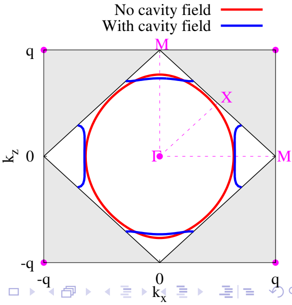
Preprints: [JK, Bhaseen, & Simons 1309.2464, Piazza & Strack 1309.2714, Chen *et al.* 1309.3784]



Dimensionless variables and free energy

- Rescale with $\sqrt{2}q, \omega_r = \hbar^2 q^2/2m$, Dimensionless variables:
  - $n_F = N/N_L$
  - $\omega \to \tilde{\omega}$
  - $\Omega \to \eta$
  - $\langle\psi\rangle \to \phi$
- Free energy $f = F/N_L\omega_r$

$$f(\tilde{\omega}, \eta, n_F \to \mu; \phi) = \tilde{\omega}\phi^2 + \mu n_F - \frac{1}{\beta}\int_{BZ} d^2k \sum_n \ln\left[1 + e^{-\beta(\epsilon_{\mathbf{k},n} - \mu)}\right]$$

- $\epsilon_{\mathbf{k},n}$ from $\hat{h} = -\nabla^2 - V(\eta, \phi; \mathbf{r})$
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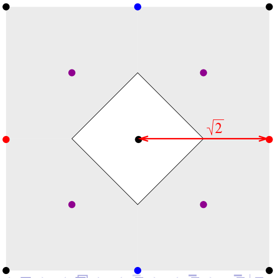

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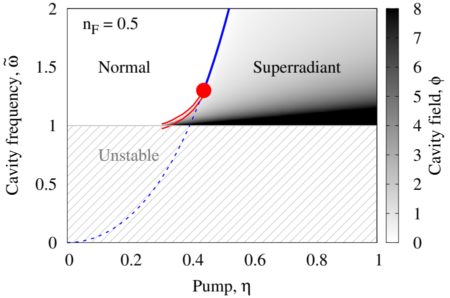

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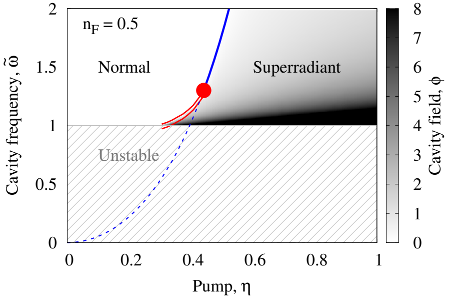

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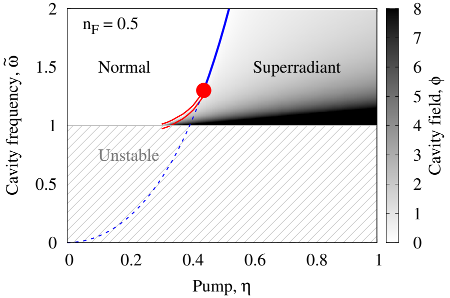

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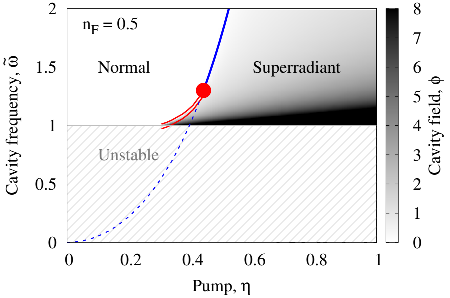
$$f = a\phi^2 + b\phi^4 + c\phi^6$$

$b < 0$ at small η.

Origin of first order transition



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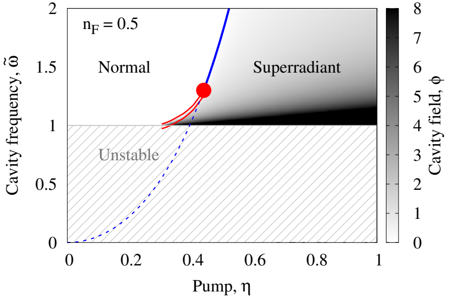
Origin of first order transition



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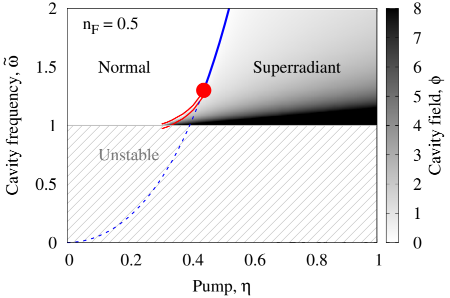

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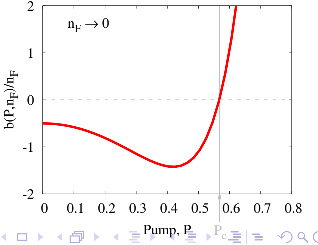

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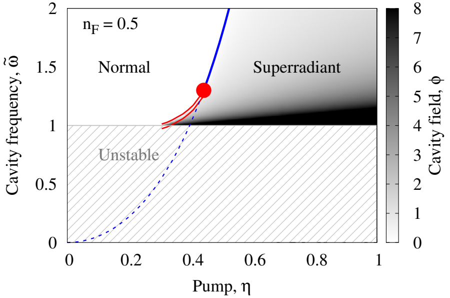

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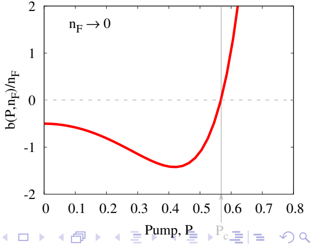
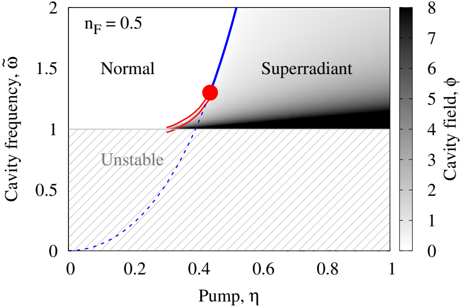


Higher fillings

$$f = a\phi^2 + b\phi^4 + c\phi^6$$

- Phase diagram unchanged for $n_F < 1$
- 2nd order line $a = 0$
- Tricritical ● at $a = b = 0$



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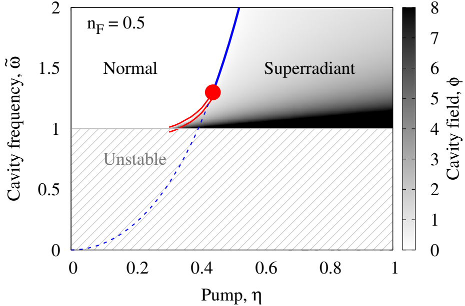

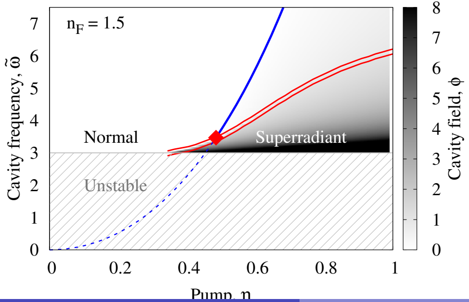

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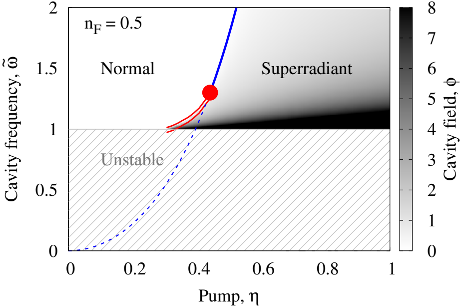

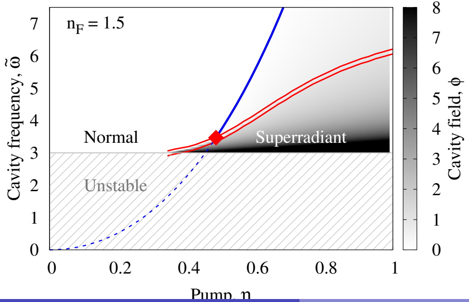

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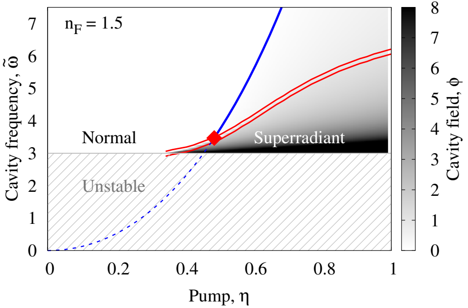

Liquid–gas transition

- $f(\phi) \rightarrow$ multiple minima
- Higher orders in ϕ

- Plot bands $\inf_k[\epsilon_{k,n}]$
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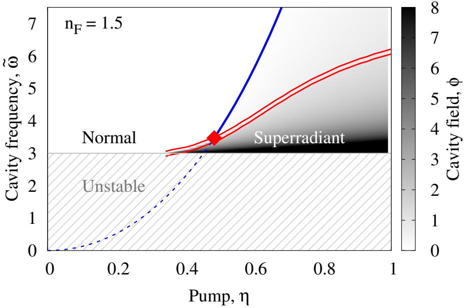

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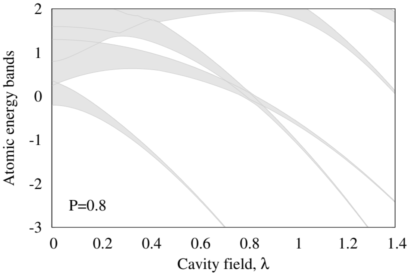

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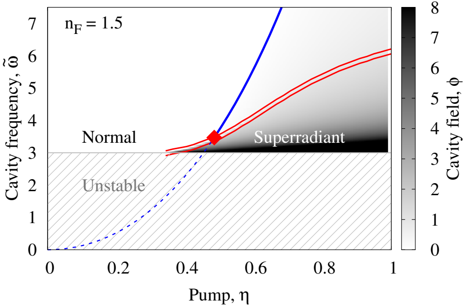

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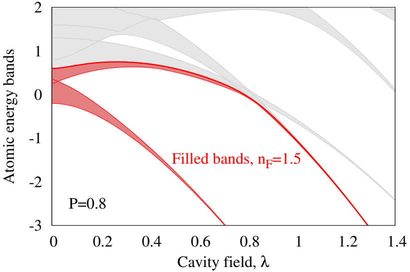

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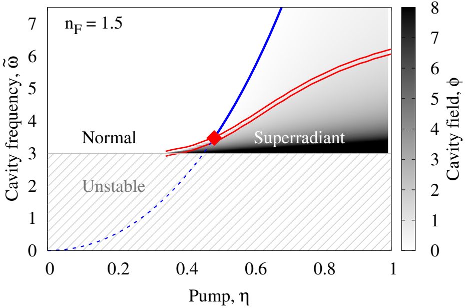

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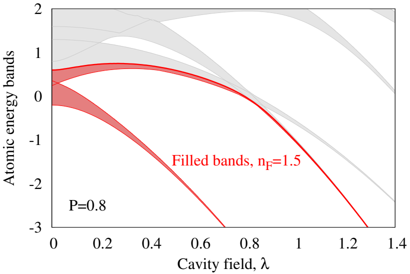
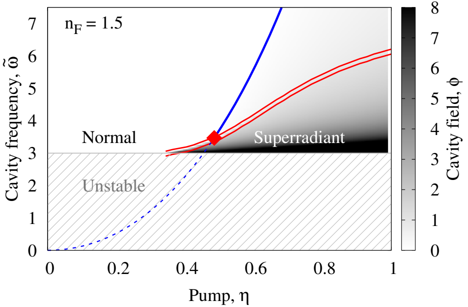
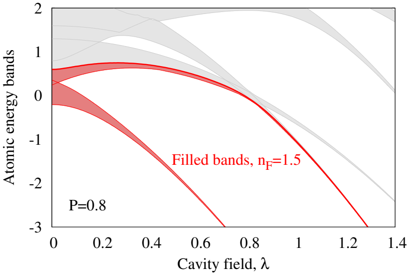
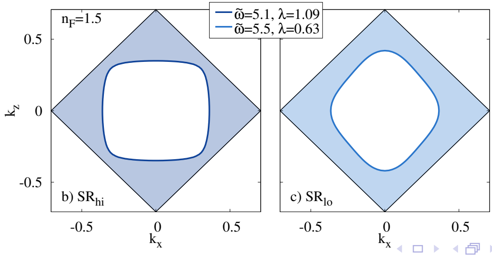


Change of Fermi surface



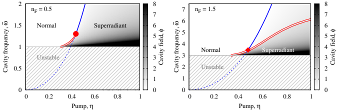


Near half-filled 2nd band — FS distortion



Phase diagram vs density

- Phase topology change:



- Fix η, plot vs n_F
- SR–SR after critical point ○
- Peak in 2nd order line $0 = a(\tilde{\omega}, n_F, \eta) = \tilde{\omega} + \chi(\eta, n_F)$
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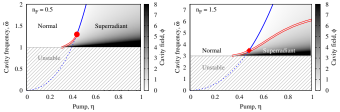

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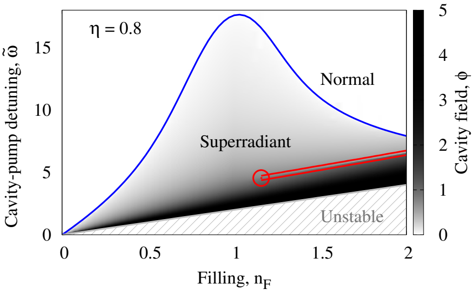

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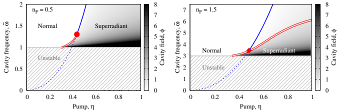

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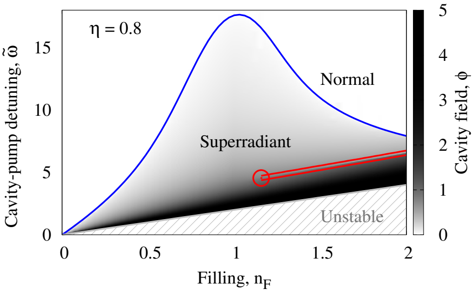

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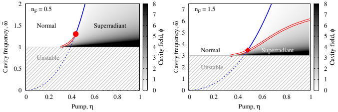

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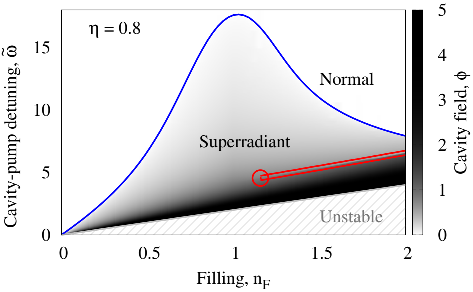

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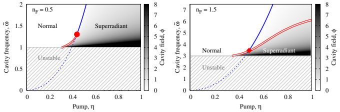

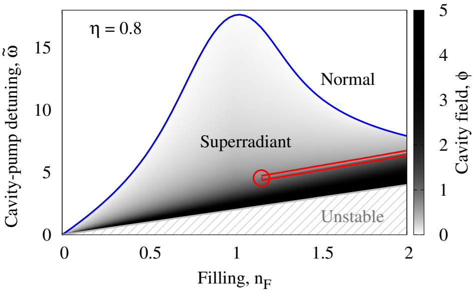

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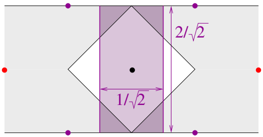

Outline

Open system vs ground state phase diagram

- Open system, $\dot{\rho} = -i[H, \rho] - \kappa\mathcal{L}[\psi]$
- Instead of $\min(F) \rightarrow$ stable attractors
- What survives — Normal-SR boundary
  - Fluctuations $\delta\psi = ue^{-i\nu t} + v^* e^{i\nu^* t}$, obey $\mathbf{M}(\nu)\begin{pmatrix} u \\ v \end{pmatrix} = 0$

    Stable if $Im[\nu] > 0$.
- What must change
  - Unstable region $\rightarrow$ new attractors
- Known unkowns
  - First order transitions/multistability?
  - Spinodal lines?
  - Limit cycles?

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- Cannot recover occupations

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Classical EOM
($|\mathbf{S}| = N/2 \gg 1$)

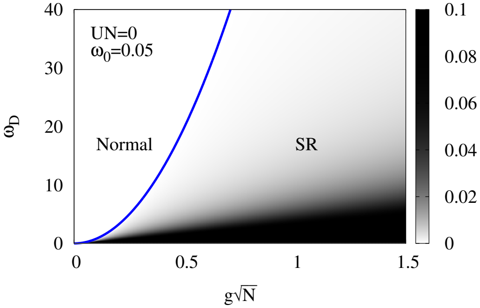
$$\begin{aligned}\dot{S}^- &= -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^*)S^z \\ \dot{S}^z &= ig(\psi + \psi^*)(S^- - S^+) \\ \dot{\psi} &= -[\kappa + i(\omega + US^z)]\psi - ig(S^- + S^+)\end{aligned}$$

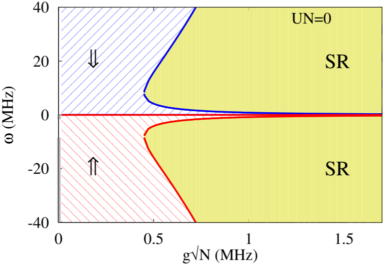
- Recover Retarded Green's function (spectrum)
- Cannot recover occupations

Long-time behaviour:

- Fixed points: $\dot{\mathbf{S}} = 0, \dot{\psi} = 0$
- Limit cycles?

Equilibrium Dicke vs open phase diagram, $UN = 0$





- Shift boundary $(\kappa^2 + \omega^2)/\omega = -\chi(\omega_0)$
- Allow negative $\omega \rightarrow$ inverted

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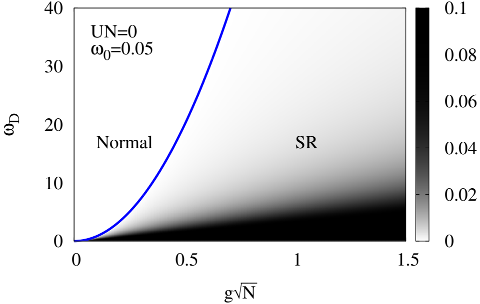

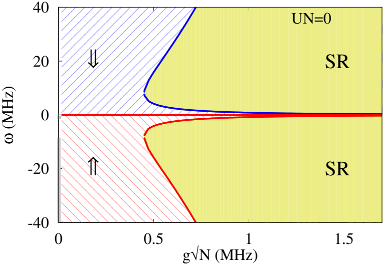

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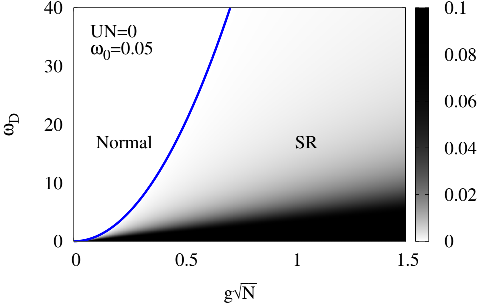

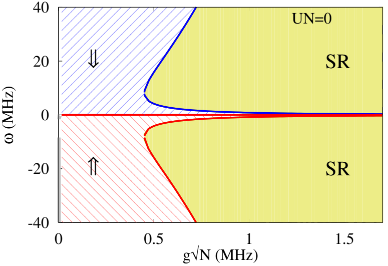

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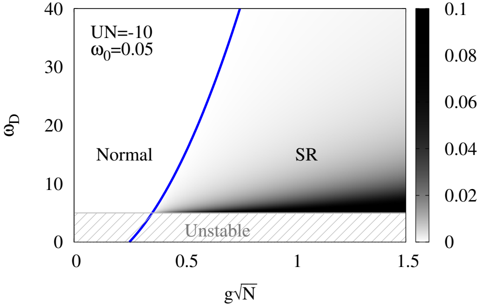
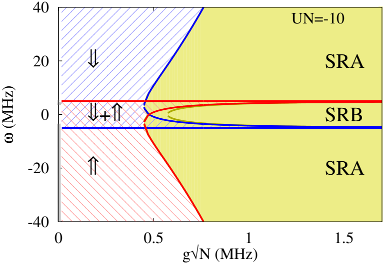
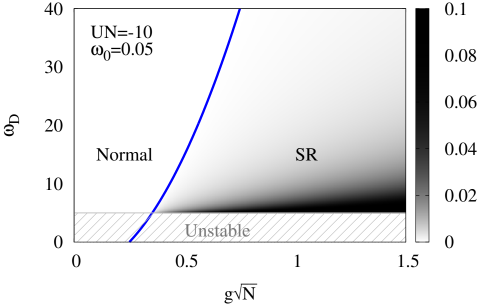
... Dicke ... $UN = -10\text{MHz}$

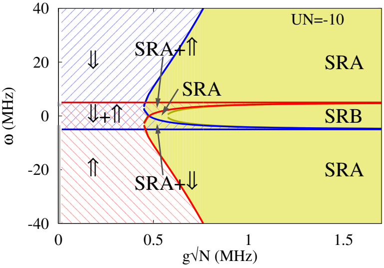




- Coexistence regions
- Unstable → SRB

. . . Dicke . . . $UN = -10\text{MHz}$





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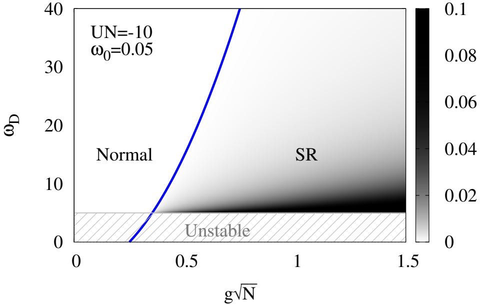

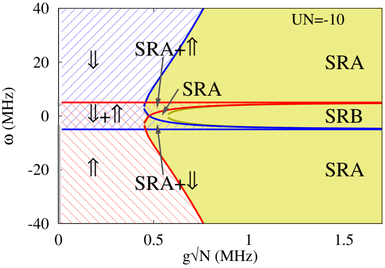

- Coexistence regions
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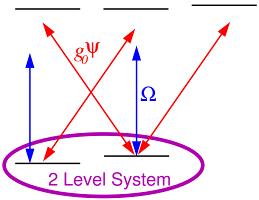
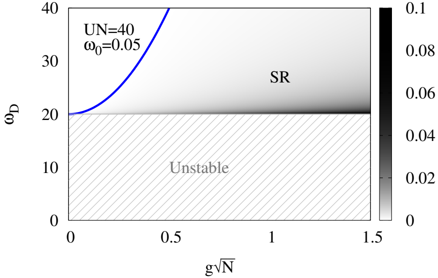
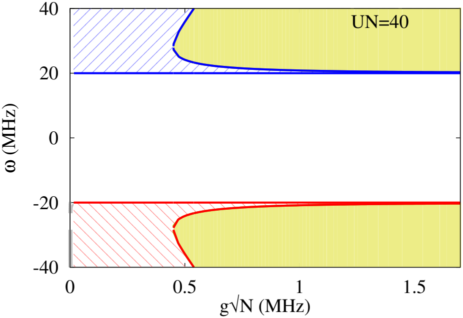
. . . Dicke . . . $UN = +40$MHz

Changing U:







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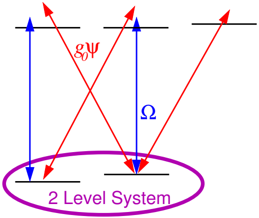

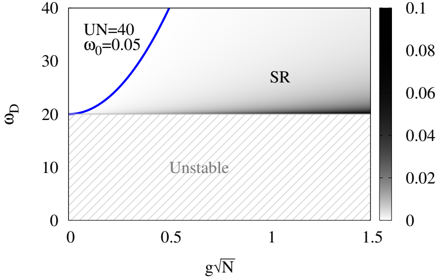

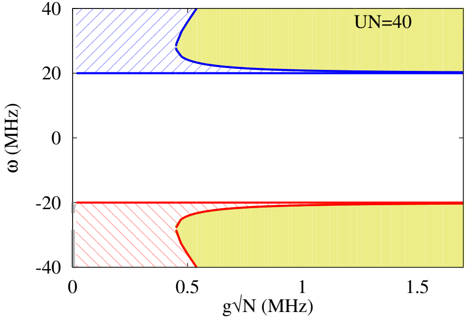

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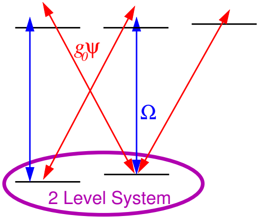
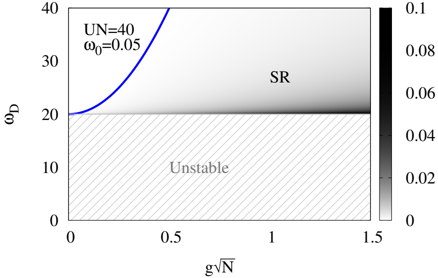
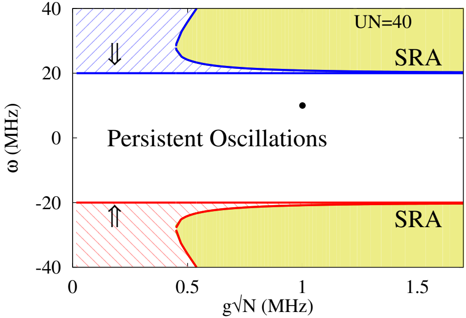
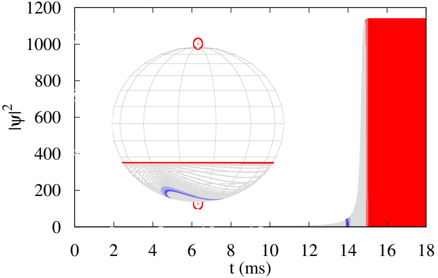
Changing *U*:



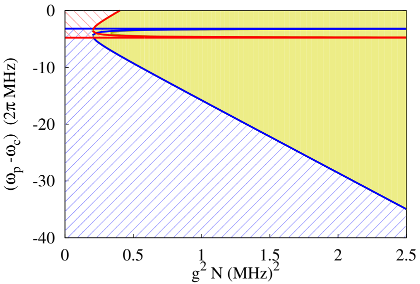


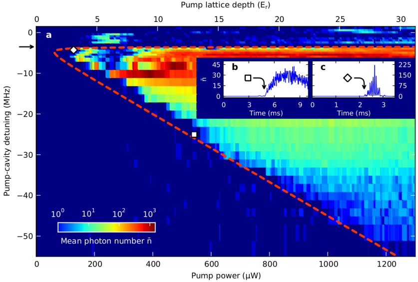




Outline

Comparison to experiment: $UN = -10\text{MHz}$



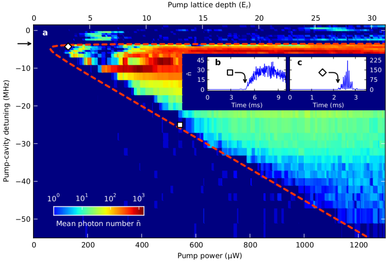


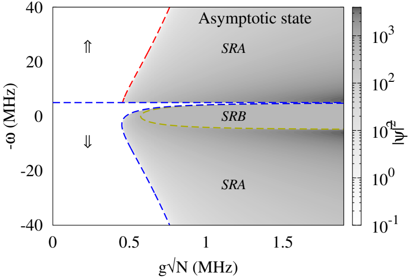
$UN = -10\text{MHz}$

Adapted from: [Bhaseen *et al.* PRA '12]

[Baumann *et al* Nature '10]

Timescale to reach steady state





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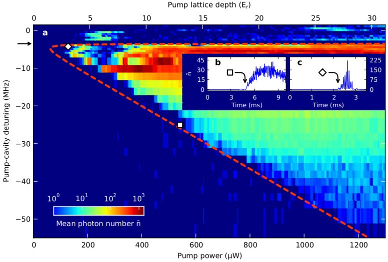

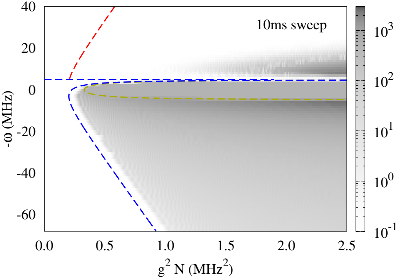

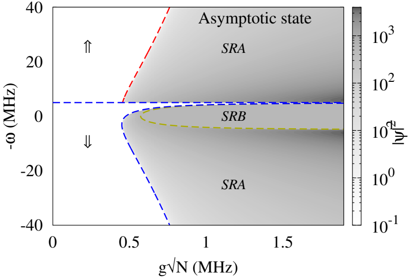

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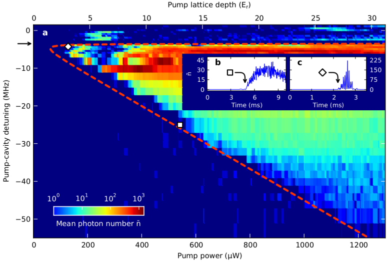
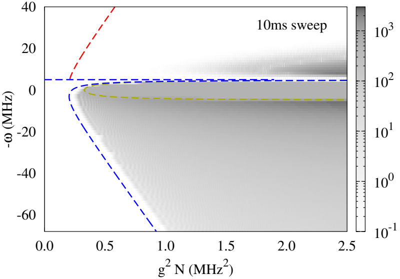
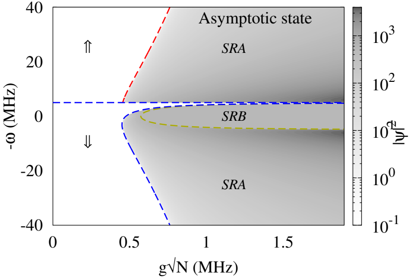
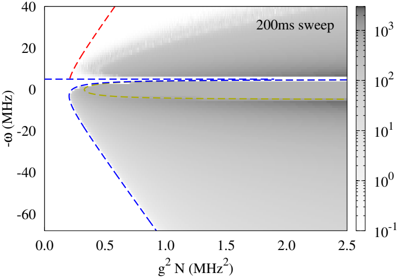


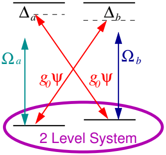






Timescales for dynamics: Why so slow and varied?

Suppose co- and counter-rotating terms differ



$$H = \ldots + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + \ldots$$

- SR(A) near phase boundary at small $\delta g \rightarrow$ Critical slowing down
- SR(A), SR(B) continuously connect

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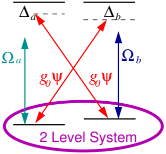

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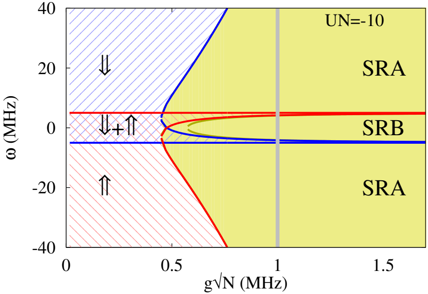

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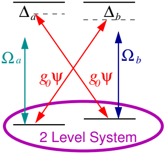

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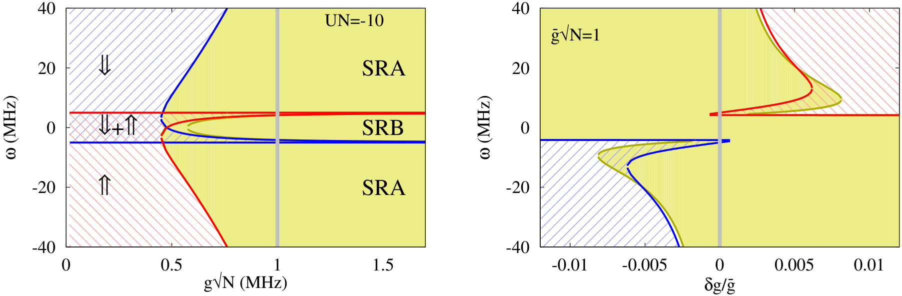

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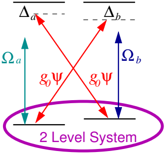

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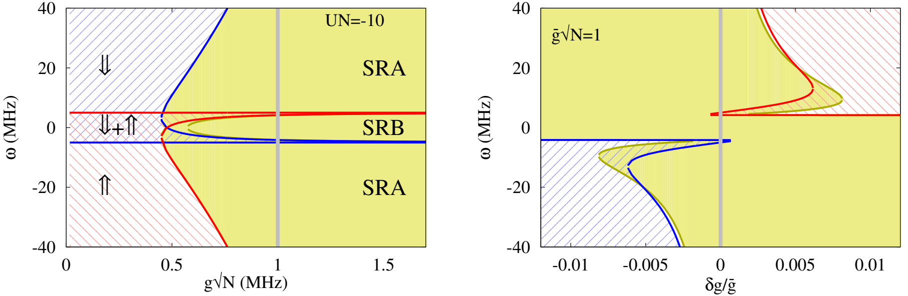

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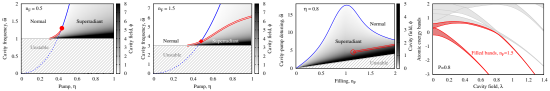
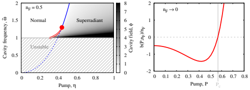
Summary

- Fermions self organisation, liquid gas, and multicritical points

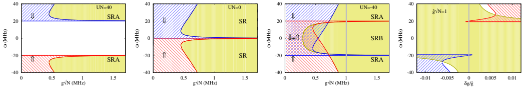


- First order transitions for bosons, outside Dicke model



JK, Bhassen, Simons *et al.* `arXiv:1309.2464`

- Dicke model shows many dynamical phases

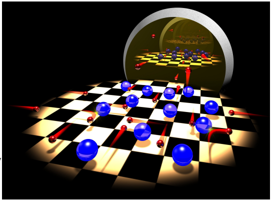


JK *et al.* PRL '10, Bhaseen *et al.* PRA '12

Many body quantum optics and correlated states of light

9:00 am on Monday 28 October 2013 – 5:00 pm on Tuesday 29 October 2013

at:The Royal Society at Chicheley Hall, home of the Kavli Royal Society International Centre, Buckinghamshire



Theo Murphy international scientific meeting organised by Dr Jonathan Keeling, Professor Steven Girvin, Dr Michael Hartmann and Professor Peter Littlewood FRS.

List of speakers and chairs

Professor Iacoppo Carusotto, Professor Andrew Cleland, Professor Hui Deng, Professor Tilman Esslinger, Professor Rosario Fazio, Professor Ed Hinds, Professor Andrew Houck, Professor Ataç İmamoğlu, Professor Jens Koch, Professor Misha Lukin, Professor Martin Plenio, Professor Arno Rauschenbeutel, Professor Timothy Spiller, Professor Jacob Taylor, Professor Hakan Tureci, Professor Andreas Wallraff

Attending this event

This is a residential conference which allows for increased discussion and networking. It is free to attend, however participants need to cover their accommodation and catering costs if required. Places are limited and therefore pre-registration is essential.

Classical dynamics of the extended Dicke model

Open dynamical system:

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi + \psi^\dagger)(S^- + S^+) + U S_z \psi^\dagger\psi.$$

$$\partial_t \rho = -i[H,\rho] - \kappa(\psi^\dagger\psi\rho - 2\psi\rho\psi^\dagger + \rho\psi^\dagger\psi)$$

- Neglects quantum fluctuations
- Linearisation about fixed point $\rightarrow$ stability, spectrum

[JK *et al.* PRL '10, Bhaseen *et al.* PRA '12]

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Fixed points (steady states)

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- $\psi = 0, \mathbf{S} = (0, 0, \pm N/2)$ always a solution.
- If $g > g_c$, $\psi \neq 0$ too
  - A $S^y = -\Im[S^-] = 0$
  - B $\psi' = \Re[\psi] = 0$

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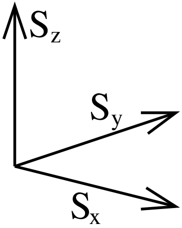

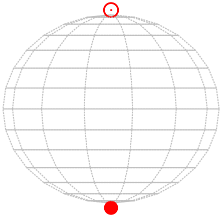

Small g: $\Uparrow, \Downarrow$ only.
($\omega = 30\text{MHz}$, $UN = -40\text{MHz}$)

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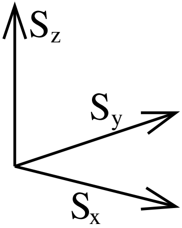

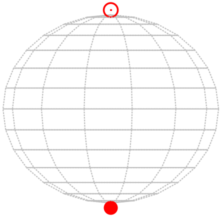

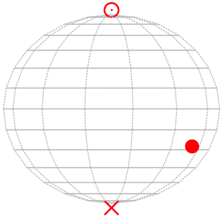

Small g: ⇑, ⇓ only. Larger g: SR too.
($\omega = 30$MHz, $UN = -40$MHz)

Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

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Two-level systems — dipole-dipole coupling

$$H = \omega_0 S^z + \omega \psi^\dagger \psi + g(S^+ + S^-)(\psi + \psi^\dagger) + N\zeta(\psi + \psi^\dagger)^2 - \eta(S^+ - S^-)^2$$

(nb $g^2, \zeta, \eta \propto 1/V$).

Ferroelectric transition

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Gauge transform to dipole gauge $\mathbf{D} \cdot \mathbf{r}$

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + \bar{g}(S^+ - S^-)(\psi - \psi^\dagger)$$

"Dicke" transition at $\omega_0 < N\bar{g}^2/\omega \equiv 2\eta N$

But, ψ describes **electric displacement**

Grand canonical ensemble

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- If $H \rightarrow H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)|\omega_0 - \mu|$
- Fix density / fix $\mu > 0$ — pumping

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- μ hits lowest mode

[Eastham and Littlewood, PRB '01]

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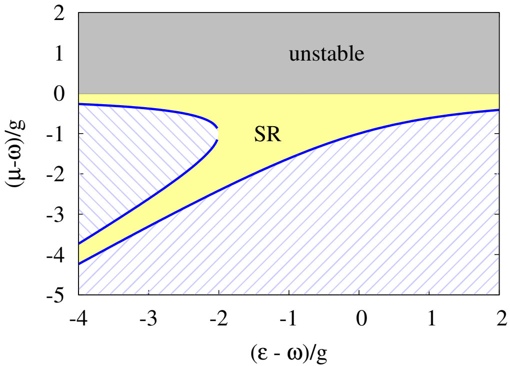

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