Condensation vs lasing and superfluidity of coupled light-matter systems

Jonathan Keeling

ICTP, Trieste, December 2012

Acknowledgements

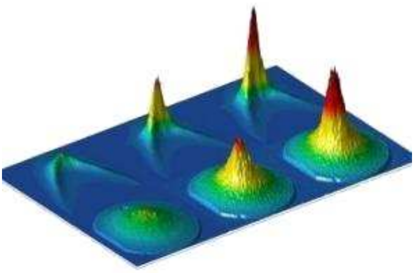
People:

Funding:

EPSRC
Engineering and Physical Sciences
Research Council

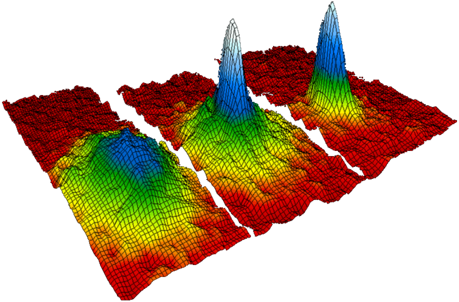
Bose-Einstein condensation: macroscopic occupation

Polaritons. $\sim$ 20K



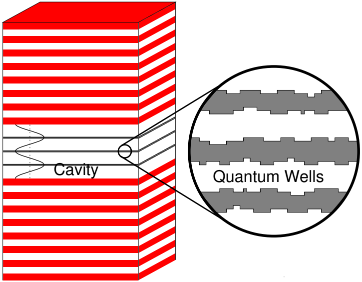
[Kasprzak *et al.* Nature, ’06]

Atoms. $\sim 10^{-7}$K



[Anderson *et al.* Science ’95]

Microcavity polaritons



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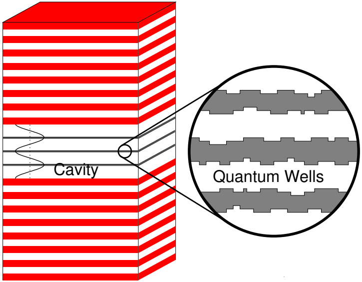
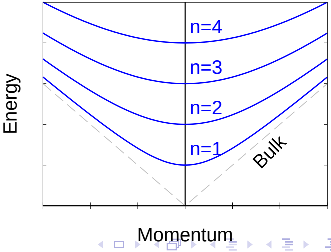


Cavity photons:

$$\omega_k = \sqrt{\omega_0^2 + c^2k^2}$$
$$\simeq \omega_0 + k^2/2m^*$$
$$m^* \sim 10^{-4} m_e$$



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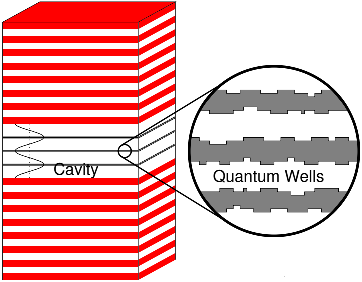

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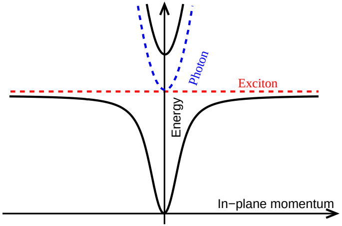

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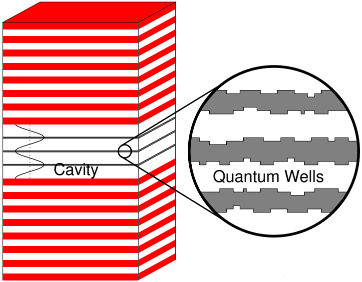

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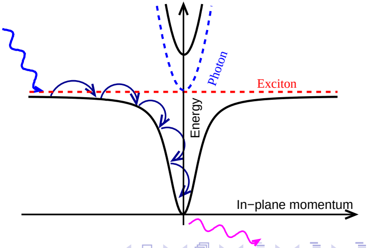

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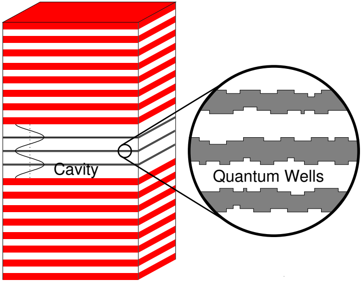

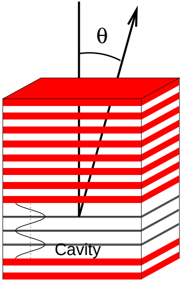

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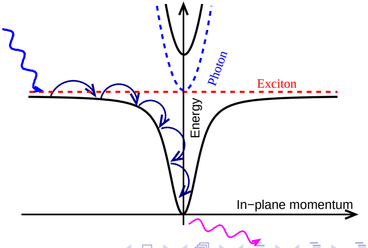
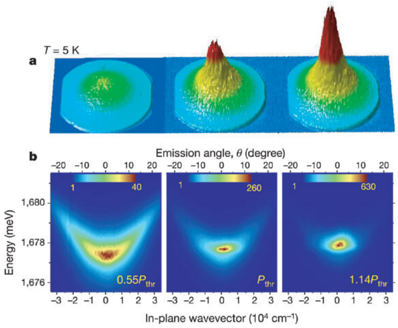
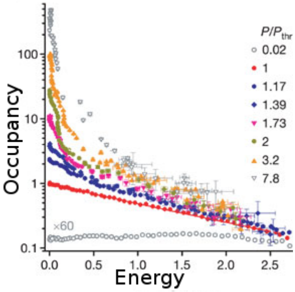


Polariton experiments: occupation and coherence





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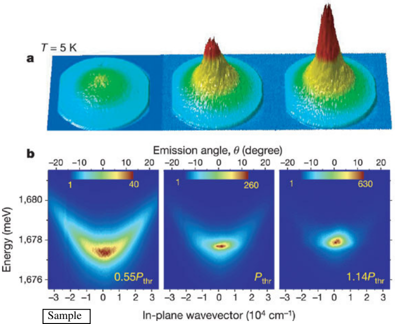
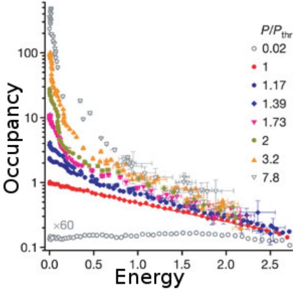
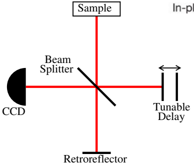
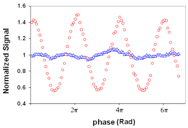
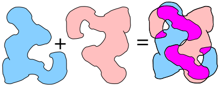


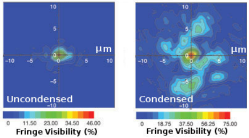






Coherence map:

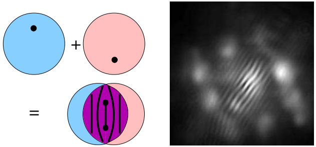




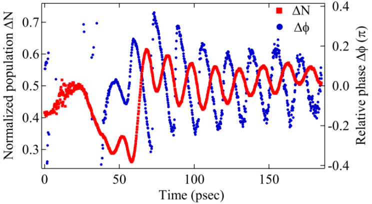
[Kasprzak, *et al.* Nature, ’06]

(Some) other polariton condensation experiments

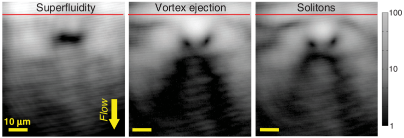
- Quantised vortices
  [**Lagoudakis *et al.* Nat. Phys. ’08**. Science ’09, PRL ’10; Sanvitto *et al.* Nat. Phys. ’10; Roumpos *et al.* Nat. Phys. ’10]



- Josephson oscillations
  [**Lagoudakis *et al.* PRL ’10**]



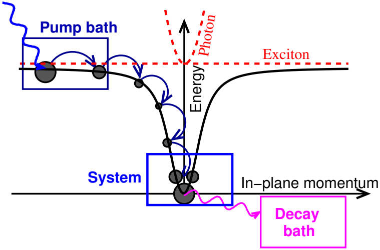
- Pattern formation/Hydrodynamics
  [**Amo *et al.* Science ’11**, Nature ’09; Wertz *et al.* Nat. Phys ’10]



Non-equilibrium approach: Steady state, and fluctuations

$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}},$$

$$H_{\text{sys}} = \sum_k \omega_k \psi_k \psi_k^\dagger + \sum_\alpha g_\alpha (\phi_\alpha^\dagger \psi_k + \text{H.c.}) + H_{\text{ex}}[\phi_\alpha, \phi_\alpha^\dagger]$$



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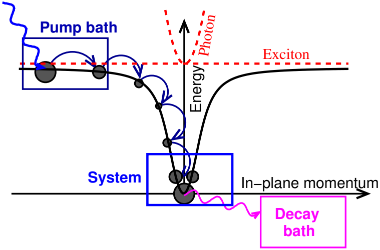

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Steady state, $\psi(\mathbf{r}, t) = \psi_0 e^{-i\mu_S t}$.

Self-consistent equation: $(i\partial_t - \omega_0 + i\kappa)\,\psi = \sum_\alpha g_\alpha \langle \phi_\alpha \rangle$

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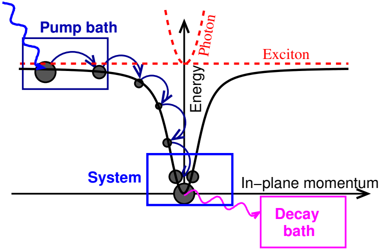

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$$[D^R - D^A](t, t') = -i \left\langle \left[\psi(t), \psi^\dagger(t')\right]_- \right\rangle$$

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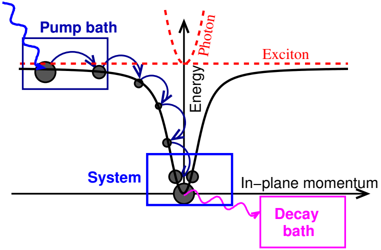

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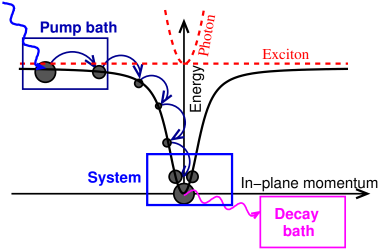

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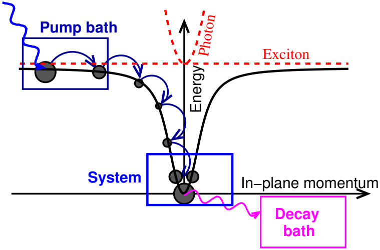

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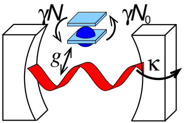
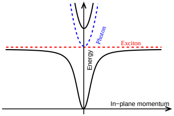
$$D^K(t, t') = -i \left\langle \left[\psi(t), \psi^\dagger(t') \right]_+ \right\rangle \qquad D^K(\omega) = (2n(\omega) + 1)\text{DoS}(\omega)$$

Lasing-condensation crossover model

- Use model that can show lasing and condensation:



$\Leftrightarrow$



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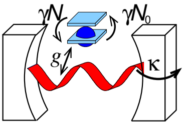

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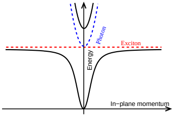

Dicke model:

$$H_{\text{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^{\dagger}_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_{\alpha} S^{z}_{\alpha} + \frac{1}{\sqrt{A}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} S^{+}_{\alpha} + \text{H.c.} \right]$$

Polariton model and equilibrium results

Localised excitons, propagating photons

$$H - \mu N = \sum_{\mathbf{k}} (\omega_{\mathbf{k}} - \mu) \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} (\epsilon_\alpha - \mu) S^z_\alpha + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{A}} \psi_{\mathbf{k}} S^+_\alpha + \text{H.c.}$$

Self-consistent polarisation and field

$$(\omega - \mu)\, \psi = \frac{1}{A} \sum_{\alpha} \frac{g_\alpha^2 \psi}{2 E_\alpha} \tanh\left(\beta E_\alpha\right), \quad {E_\alpha}^2 = \left(\frac{\epsilon_\alpha - \mu}{2}\right)^2 + g_\alpha^2 |\psi|^2$$

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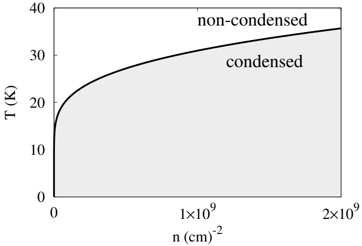

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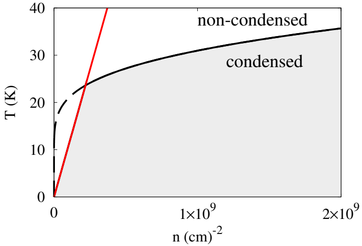

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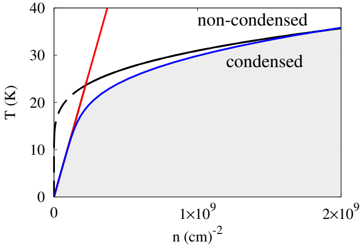

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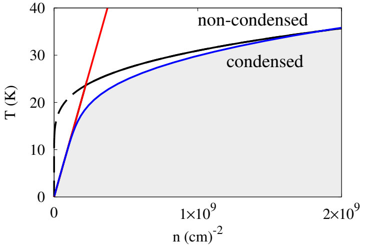
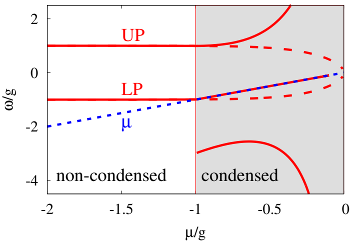
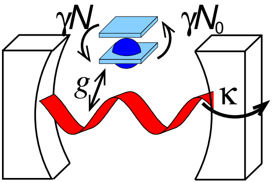


Modes (at $k = 0$)



Simple Laser: Maxwell Bloch equations

$$H = \omega\psi^\dagger\psi + \sum_\alpha \epsilon_\alpha S^z_\alpha + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{A}}\psi S^+_\alpha + \text{H.c.}$$



Maxwell-Bloch eqns: $P = -i\langle S^-\rangle, N = 2\langle S^z\rangle$

$$\partial_t\psi = -i\omega\psi - \kappa\psi + \sum_\alpha g_\alpha P_\alpha$$
$$\partial_t P_\alpha = -2i\epsilon_\alpha P_\alpha - 2\gamma P + g_\alpha\psi N_\alpha$$
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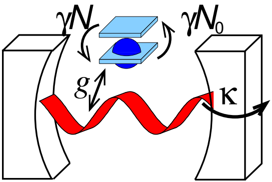

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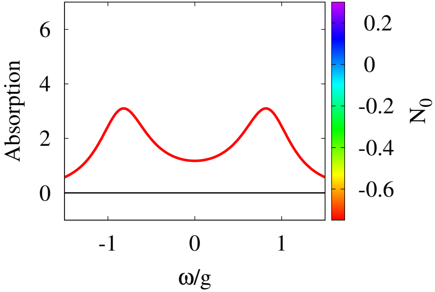

- Strong coupling. $\kappa, \gamma < g\sqrt{n}$
- Inversion causes collapse before lasing

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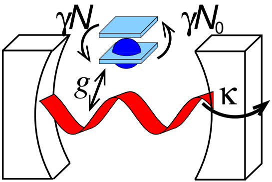

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$$\partial_t\psi = -i\omega\psi - \kappa\psi + \sum_\alpha g_\alpha P_\alpha$$
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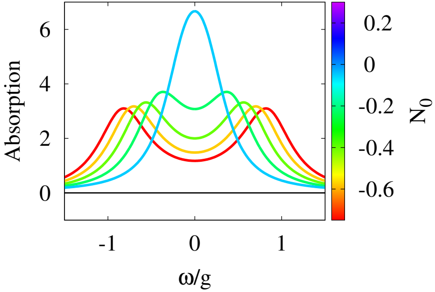
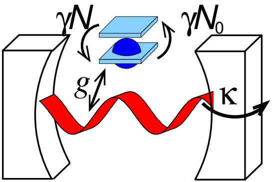


- Strong coupling. $\kappa, \gamma < g\sqrt{n}$
- Inversion causes collapse before lasing

Simple Laser: Maxwell Bloch equations

$$H = \omega\psi^\dagger\psi + \sum_\alpha \epsilon_\alpha S^z_\alpha + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{A}}\psi S^+_\alpha + \text{H.c.}$$



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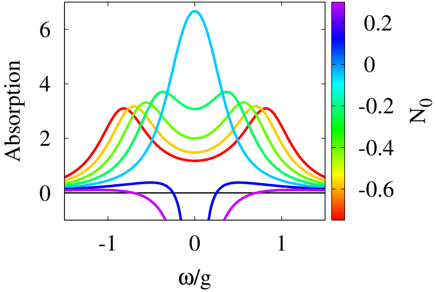

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$$\left[D^R(\nu)\right]^{-1} = \nu - \omega_k + i\kappa + \frac{g^2 N_0}{\nu - 2\epsilon + i2\gamma}$$

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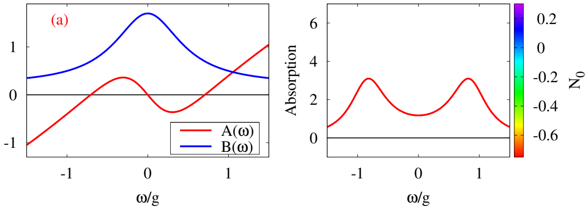

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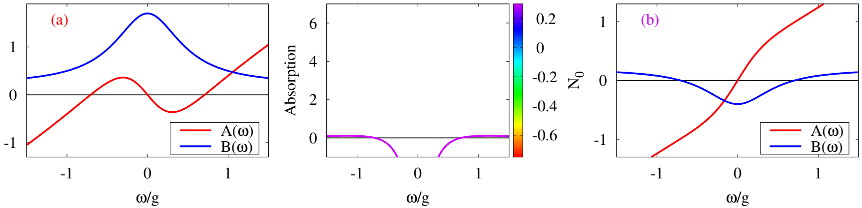

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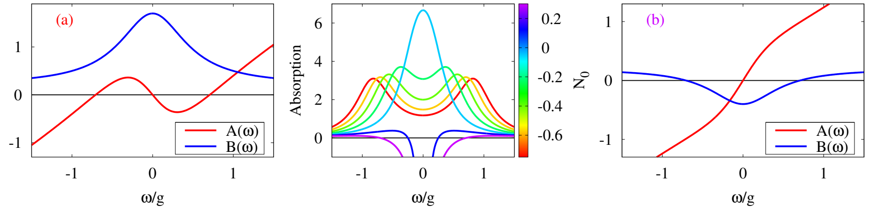

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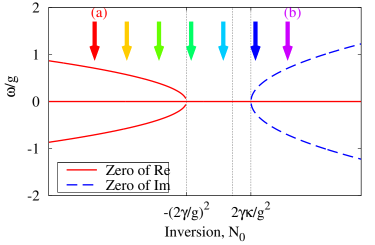

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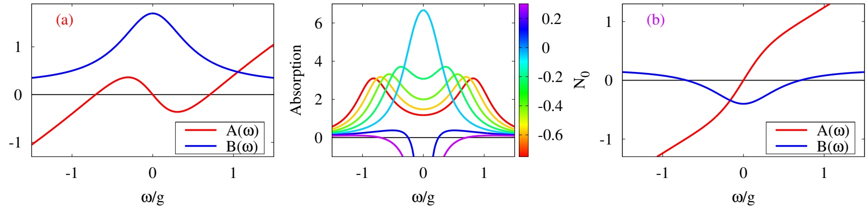

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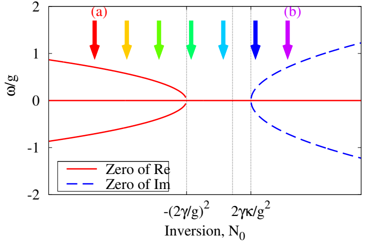
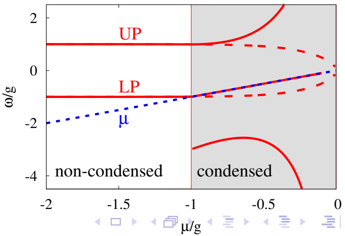
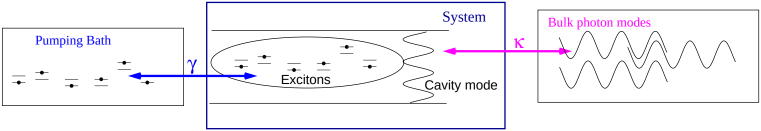
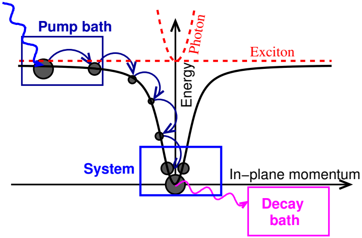


Equilibrium:



Non-equilibrium description: baths





$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

- Decay bath: Empty ($\mu \to -\infty$)
- Pump bath: Thermal μ_B, T_B

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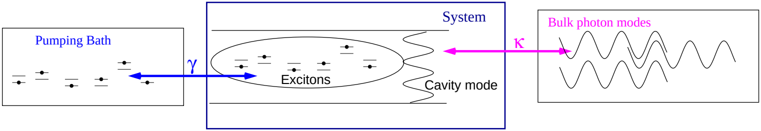

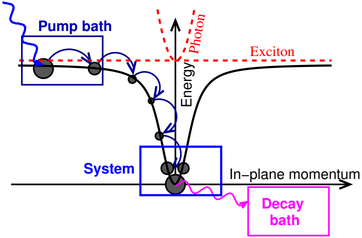

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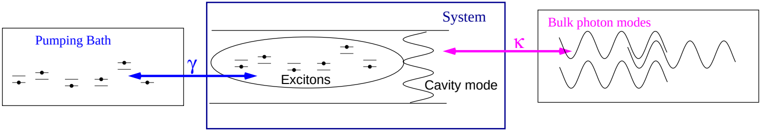

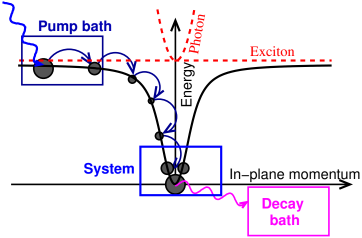

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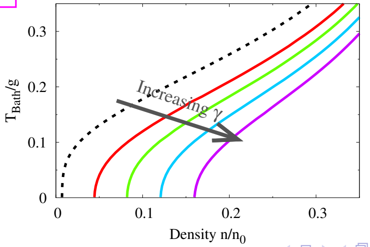
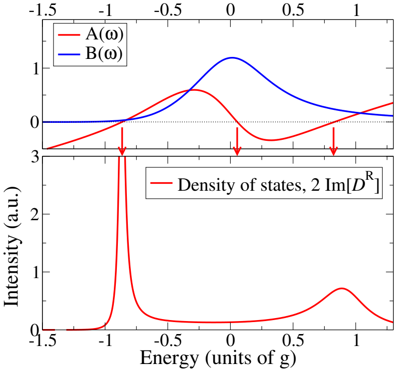
Mean field theory



Stability and evolution with pumping



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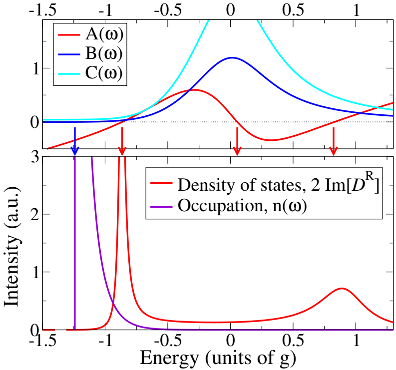

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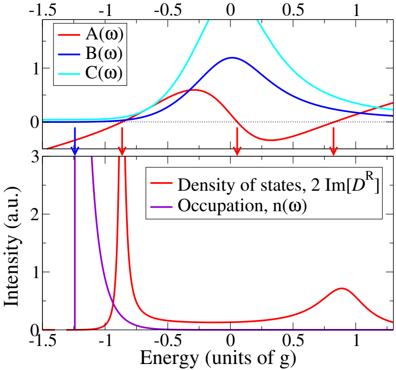

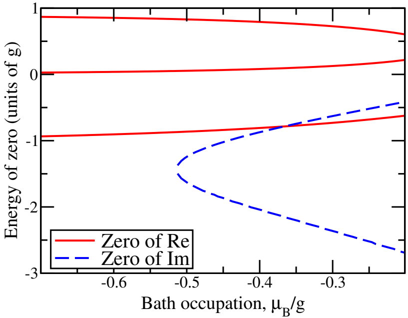

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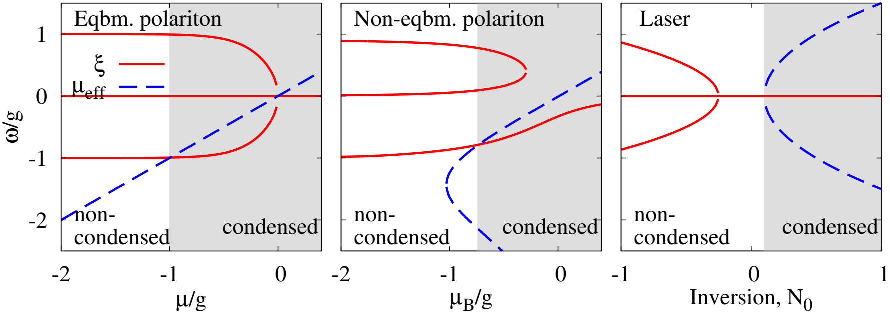
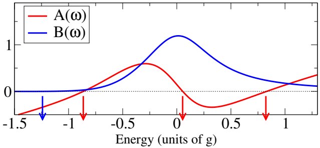
Strong coupling and lasing — low temperature phenomenon



- Laser: Uniformly invert TLS
- Non-equilibrium polaritons: Cold bath
- If $T_B \gg \gamma \to$ Laser limit



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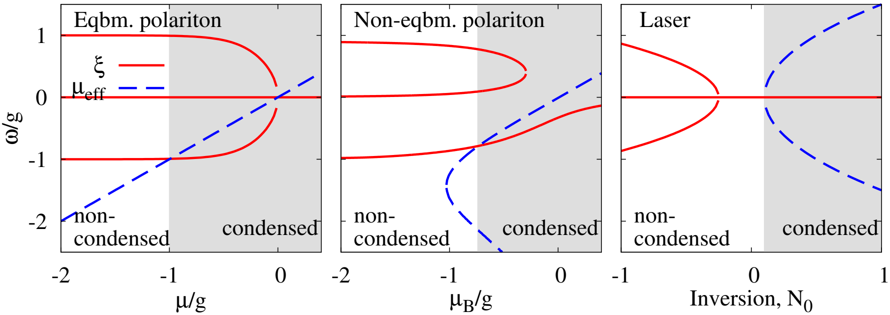

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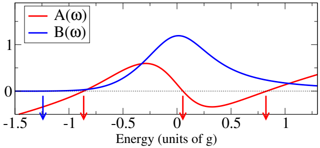

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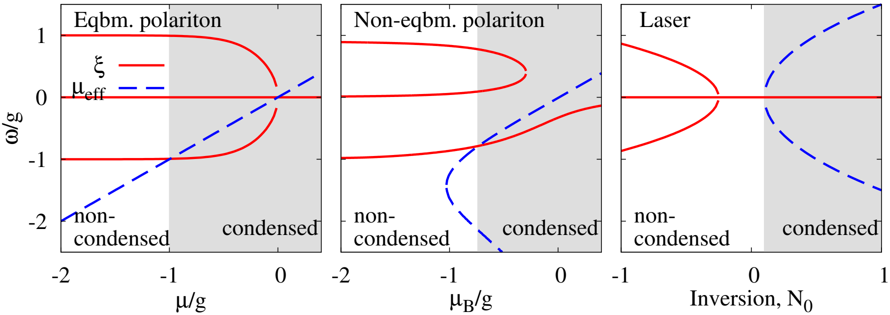

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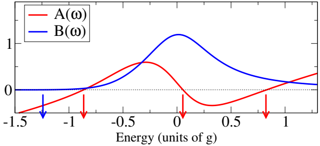
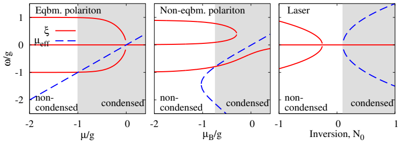


Coherence, inversion, strong-coupling

Polariton condensation:



- Inversionless
- **allows** strong coupling
- **requires** low $T \leftrightarrow$ condensation
- NB **NOT** thresholdless/single atom lasing.

- Circuit QED [Marthaler *et al.* PRL '11]
  - Noise-assisted
  - Off-resonant cavity
  - Emission/absorption $\Gamma^{\pm} \sim 2n_B(\pm\delta\omega) + 1$
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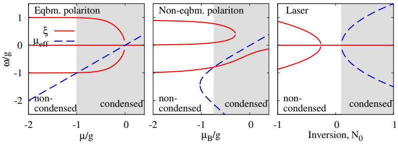

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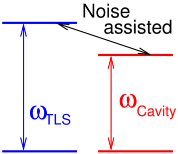

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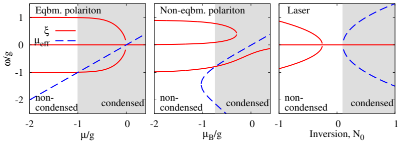

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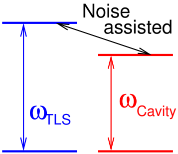

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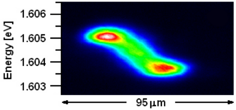
Pattern formation: 

1. Introduction to polariton condensation
2. Non-equilibrium condensation vs lasing
3. Pattern formation
4. Superfluidity

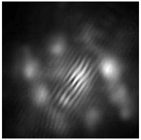
Pattern formation in experiments

Polariton Traps



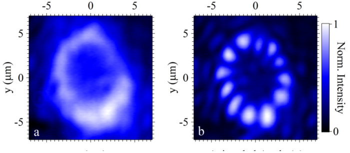
[Balili *et al.* Science '07]

Vortex formation



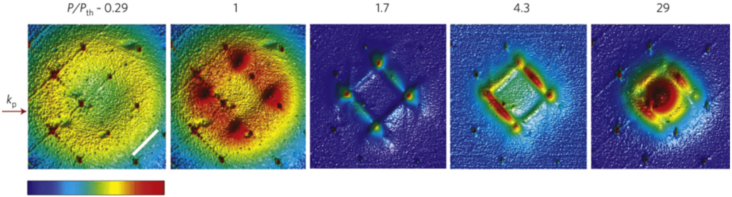
[Lagoudakis *et al.* Nat. Phys '08]

Elliptical ring pump



[Manni *et al.* PRL '11]

Patterned lattice: Momentum space image



[Kim *et al.* Nat. Phys '11]

Complex Gross-Pitaevskii equation

Steady state equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit:

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Gross-Pitaevskii equation: Harmonic trap

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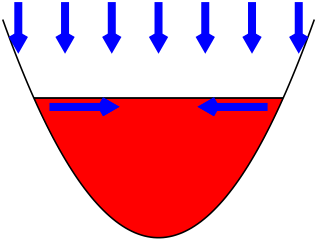

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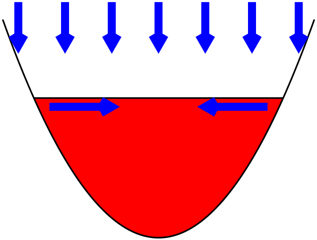
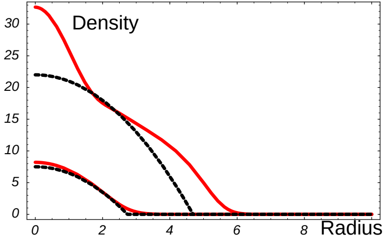
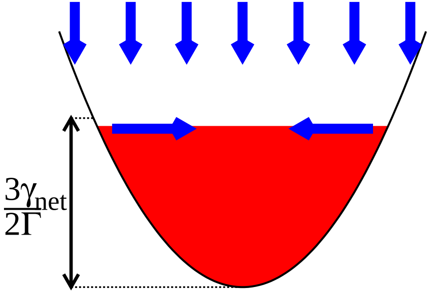




Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = (\gamma_{\text{net}} - \Gamma \rho)\rho$$



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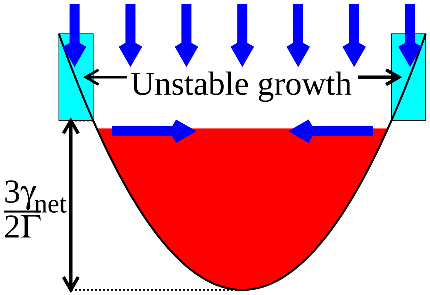

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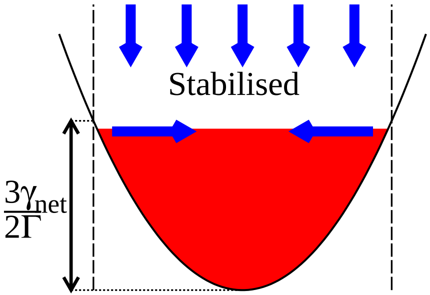

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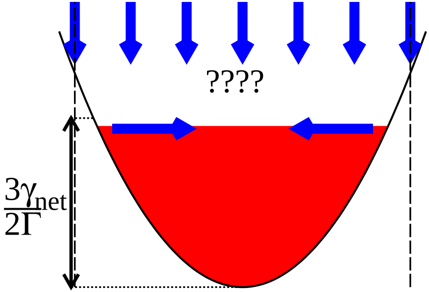
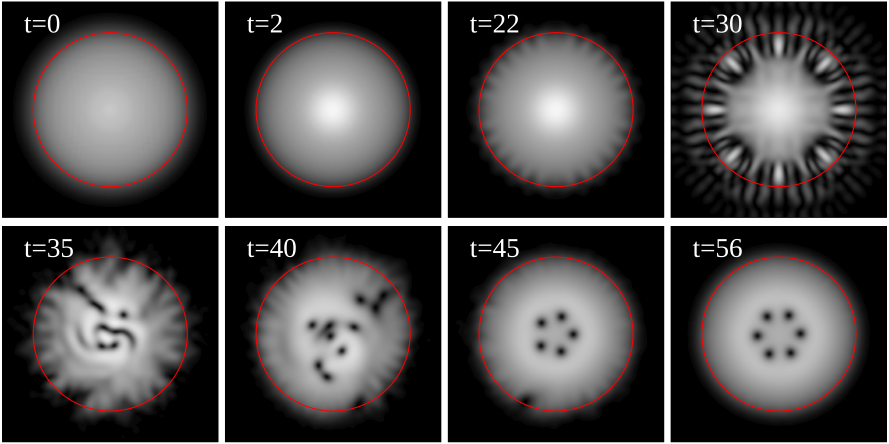
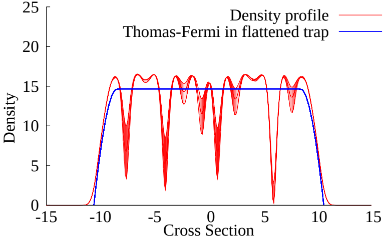


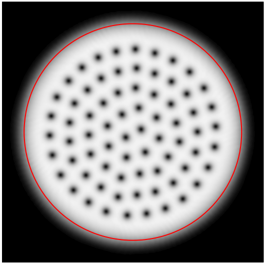
Time evolution:



[Keeling & Berloff PRL '08]

Why vortices





$$\nabla \cdot [\rho(\mathbf{v} - \Omega \times \mathbf{r})] = (\gamma_{net}\Theta(r_0 - r) - \Gamma\rho)\,\rho,$$

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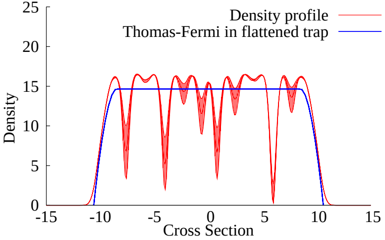

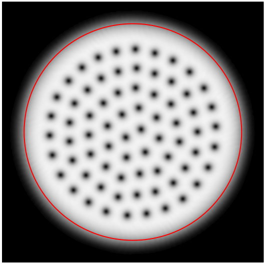

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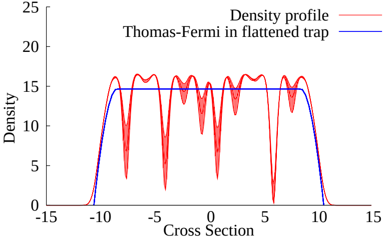

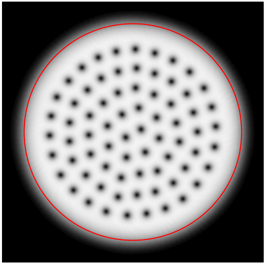

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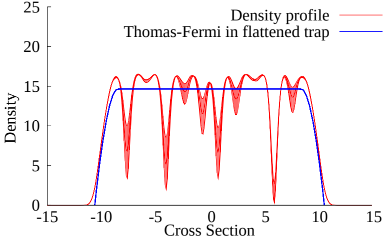

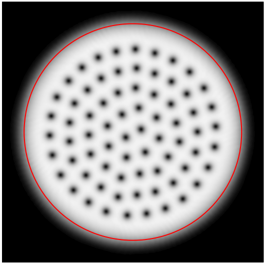

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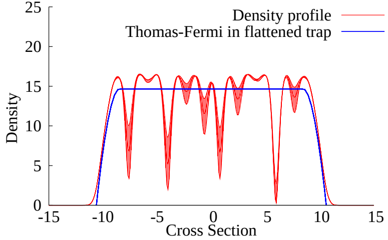

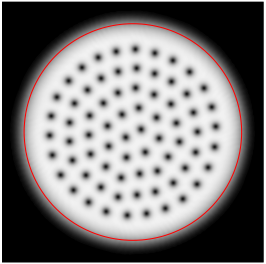

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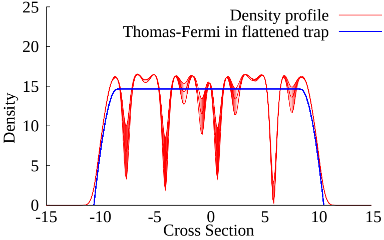

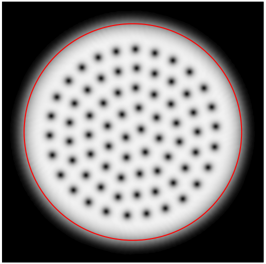

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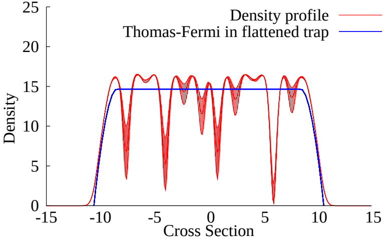

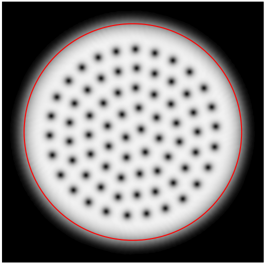

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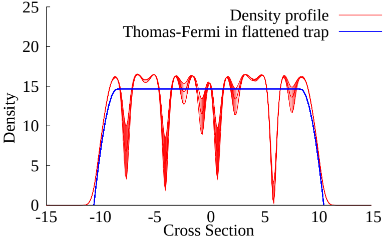

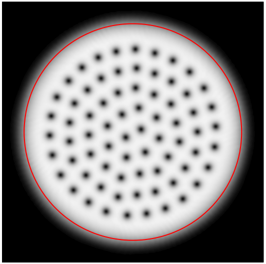

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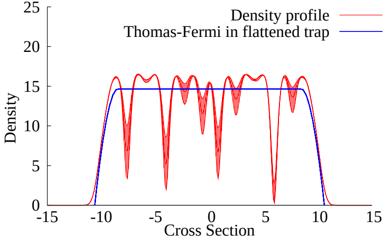

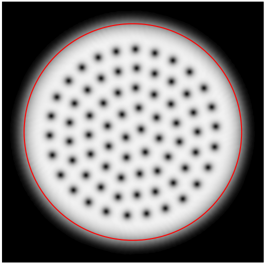

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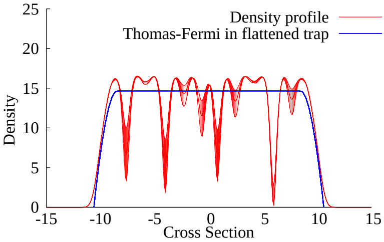

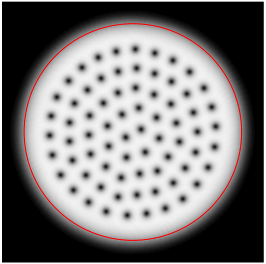

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Superfluidity

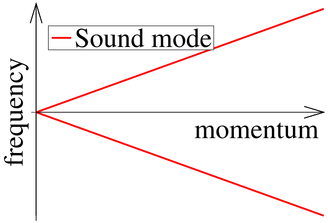
Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega, k)\right]^{-1} = \omega^2 - \xi_k^2$$

With $\xi_k \simeq ck$
Poles:

$$\omega^* = \xi_k$$



Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega,k)\right]^{-1} = (\omega + i\gamma_{\text{net}})^2 + \gamma_{\text{net}}^2 - \xi_k^2$$

With $\xi_k \simeq ck$
Poles:

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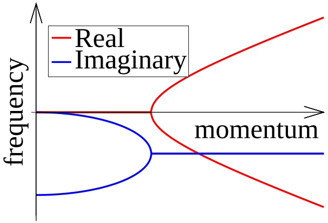
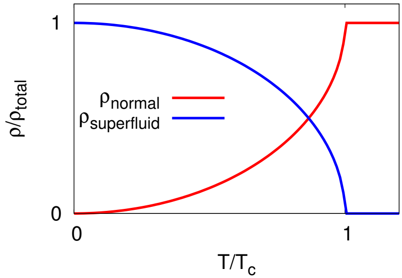


Superfluid density

- Two-fluid hydrodynamics



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- To calculate, transverse/longitudinal:

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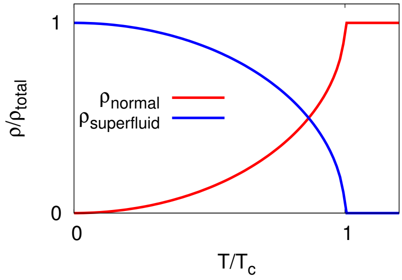

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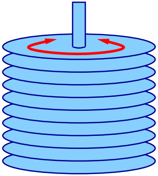

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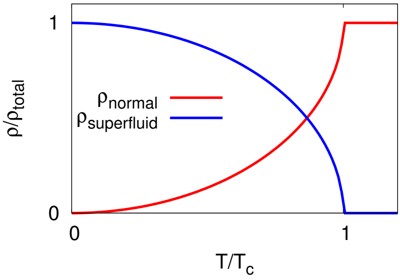

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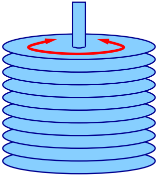

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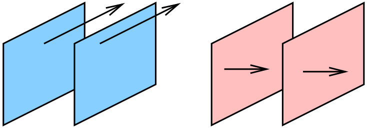

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$$\mathbf{J}(r) = \psi^\dagger i \nabla \psi$$

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- Response function:

$$H \to H - \sum_q \mathbf{f}(\mathbf{q}) \cdot \mathbf{J}_i(\mathbf{q}) \qquad J_i(\mathbf{q}) = \chi_{ij}(\mathbf{q}) f_j(\mathbf{q})$$

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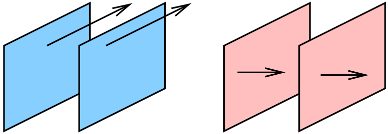

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$$\chi_{ij}(\omega = 0, \mathbf{q} \to 0) = \langle [J_i(\mathbf{q}), J_j(-\mathbf{q})] \rangle = \frac{\rho_S}{m} \frac{q_i q_j}{q^2} + \frac{\rho_N}{m} \delta_{ij}$$

Calculating superfluid response function

- Use WIDBG model
- Require vertex corrections
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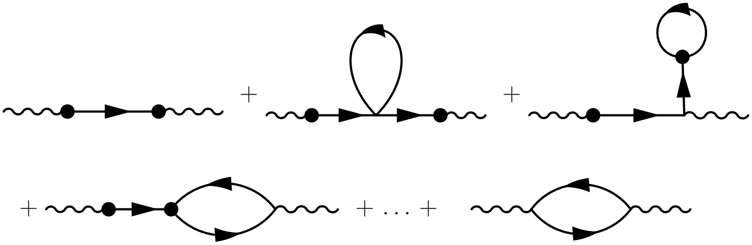

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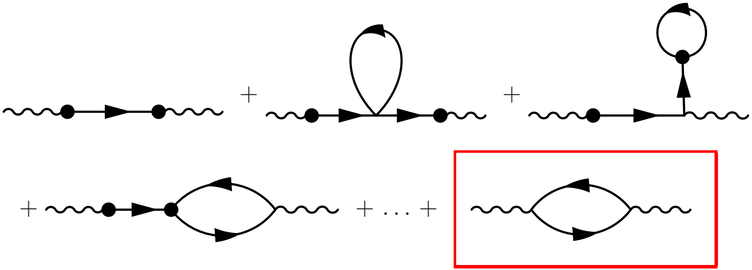

Non-equilibrium superfluid response

- Superfluid response exists because:

 $= \frac{i\psi_0 q_i}{2m}(1,-1)\,D^R(q,\omega=0)\begin{pmatrix}1\\-1\end{pmatrix}\frac{i\psi_0 q_j}{2m}$

- $D^R(\omega=0) \propto 1/\epsilon_q$ despite pumping/decay — superfluid response exists.
- Normal density:

$$\rho_N = \int d^d k\,\epsilon_k \int \frac{d\omega}{2\pi}\mathrm{Tr}\left[\sigma_z D^K \sigma_z (D^R + D^A)\right]$$

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[JK PRL '11]

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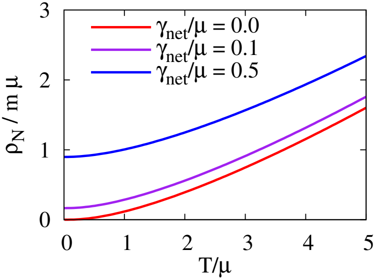
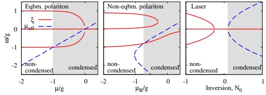


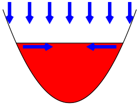
[JK PRL '11]

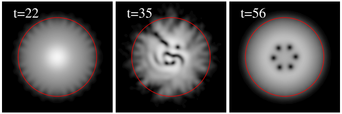
Conclusion

- Polariton condensation vs lasing

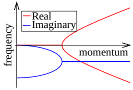


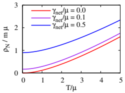
- Instability of Thomas-Fermi and spontaneous rotation





- Survival of superfluid response



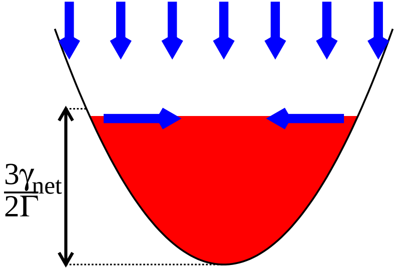


Extra slides

Instability of Thomas-Fermi: details

$$\frac{1}{2}\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = (\gamma_{\rm net} - \Gamma\rho)\rho$$

$$\partial_t\mathbf{v} + \nabla\left(U\rho + \frac{m\omega^2}{2}r^2 + \frac{m}{2}|\mathbf{v}|^2\right) = 0$$



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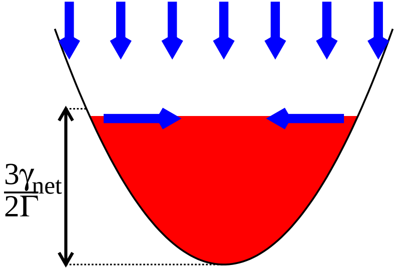

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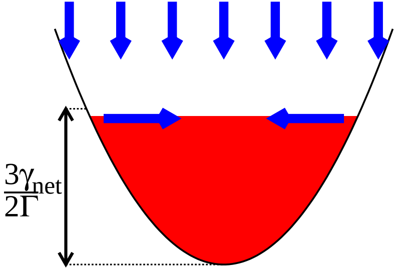

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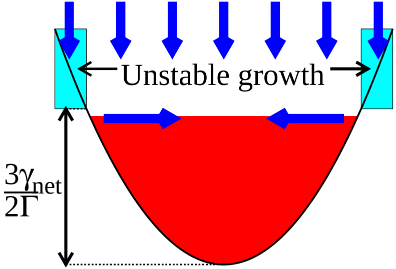

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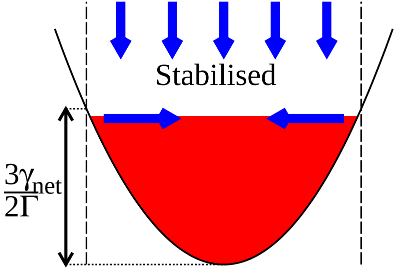

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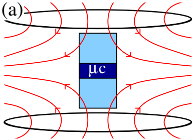
Measuring superfluid density

1. Effect rotating frame

Polariton polarization: $(\psi_{\circlearrowleft}, \psi_{\circlearrowright})$

$$H = \lambda \begin{pmatrix} \ell^2 & r^2 e^{2i\phi} \\ r^2 e^{-2i\phi} & -\ell^2 \end{pmatrix}$$



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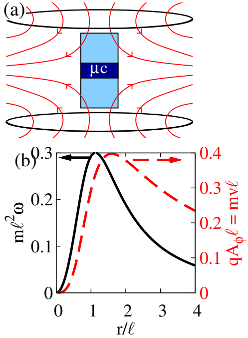
$$q\mathbf{A}_{\text{eff}} = m\omega \times \mathbf{r} = \frac{\hat{\phi}}{r}\left[1 - \frac{\ell^2}{\sqrt{r^4 + \ell^4}}\right]$$



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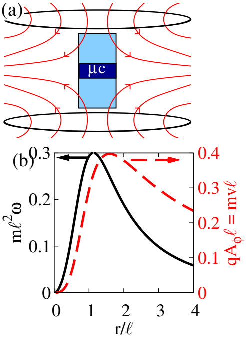
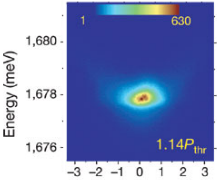


2. Measure resulting current

Energy shift of normal state:

$\Delta E = (1/2)mv^2 = 0.08/m\ell^2 \simeq 0.1\text{meV}$

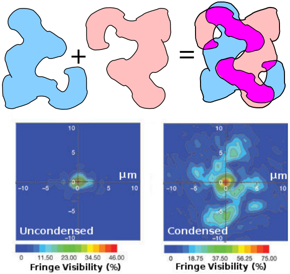


Coherence:

Coherence in a 2D Gas

Correlations:

$$g_1(\mathbf{r}, \mathbf{r}', t) = \left\langle \psi^\dagger(\mathbf{r}, t)\psi(\mathbf{r}', 0) \right\rangle$$



- $D^< = D^K - D^R + D^A$
- Generally, get:

$$\left\langle \psi^\dagger(\mathbf{r}, t)\psi(0, 0) \right\rangle \simeq |\psi_0|^2 \exp\left[-a_p \begin{cases} \ln(r/r_0) & t \simeq 0 \\ \frac{1}{2}\ln(c^2 t/\gamma_{net} r_0^2) & r \simeq 0 \end{cases} \right]$$

[Szymańska *et al.* PRL '06; PRB '07] [Wouters and Savona PRB '09]

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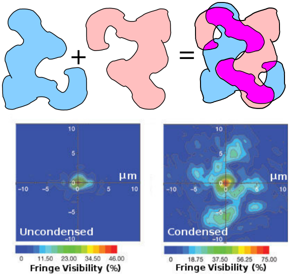

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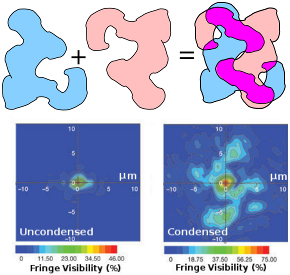

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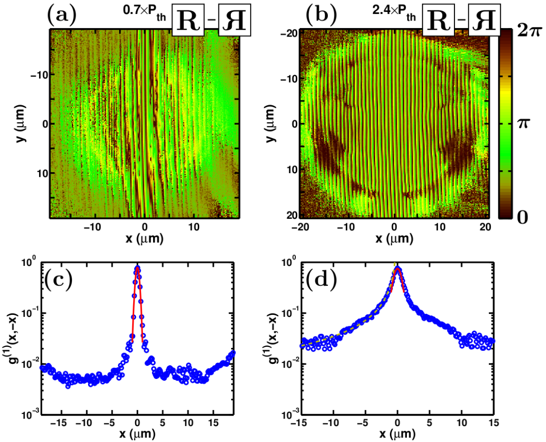

G. Rompos, *et al.* PNAS '12

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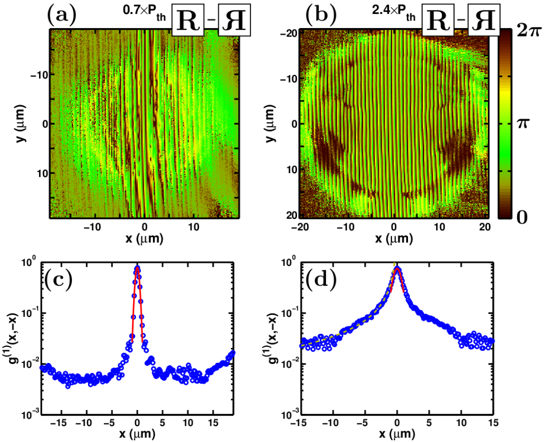
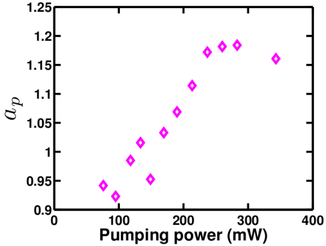


G. Rompos, *et al.* PNAS '12

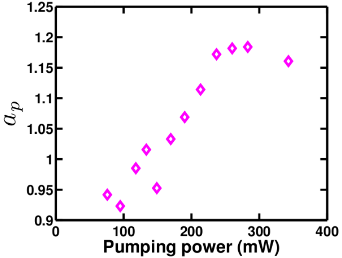
$$g_1(\mathbf{r}, -\mathbf{r}) \propto \left(\frac{r}{r_0}\right)^{-a_p}$$



Exponent in a non-equilibrium 2D gas

$$\lim_{r\to\infty} \left\langle \psi^\dagger(\mathbf{r},0)\psi(-\mathbf{r},0)\right\rangle = |\psi_0|^2 \exp\left[-D^{<}_{\phi\phi}(\mathbf{r},-\mathbf{r})\right] \propto \exp\left[-a_p \ln\left(\frac{2r}{r_0}\right)\right]$$

- Experimentally, $a_P \simeq 1.2$
- In equilibrium $a_p = \frac{mk_BT}{2\pi\hbar^2 n_s} < \frac{1}{4}$ (BKT transition)
- Non-equilibrium theory depends on thermalisation.



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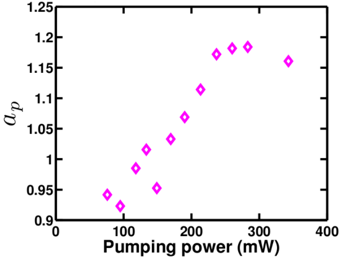

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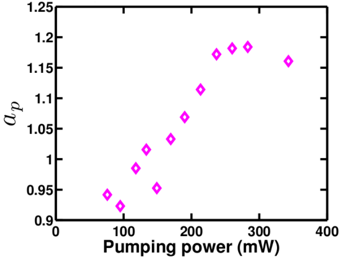

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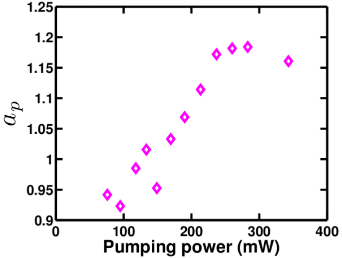

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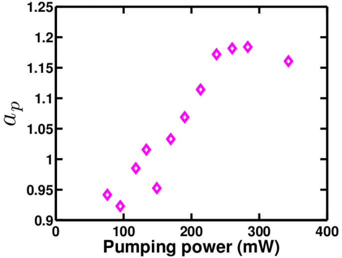

Finite size effects: Single mode vs many mode

$$\left\langle \psi^\dagger(\mathbf{r}, t)\psi(\mathbf{r}', 0)\right\rangle \simeq |\psi_0|^2 \exp\left[-D^<_{\phi\phi}(\mathbf{r}, \mathbf{r}', t)\right]$$

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$$D^<_{\phi\phi}(\mathbf{r},\mathbf{r},t) \propto \sum_n^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(\mathbf{r})|^2(1-e^{i\omega t})}{\left|(\omega + i\gamma_{\text{net}})^2 + \gamma_{\text{net}}^2 - \xi_n^2\right|^2}$$

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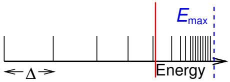
$$\Delta\xi \ll \sqrt{\frac{\gamma_{\text{net}}}{t}} \ll E_{\text{max}}$$



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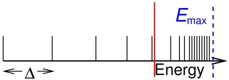

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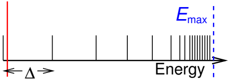

$$D^<_{\phi\phi} \sim \left(\frac{\pi C}{2\gamma_{\rm net}}\right)\left(\frac{t}{2\gamma_{\rm net}}\right)$$

(Recovers Schawlow-Townes laser linewidth)