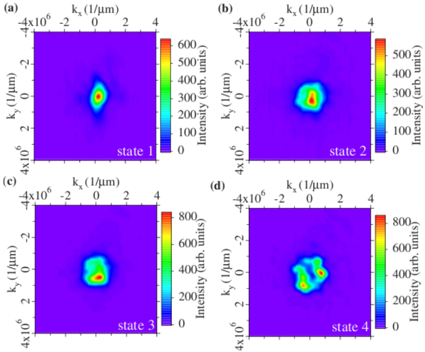
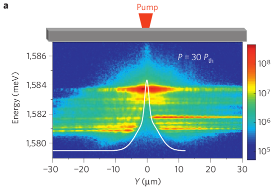
Superfluidity and pattern formation in non-equilibrium polariton condensates

Jonathan Keeling

PLMCN11, Berlin, April 2011

Motivation: Non-equilibrium features





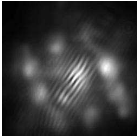
Flow from pumping spot
[Wertz *et al.*, Nat. Phys. (2010)]

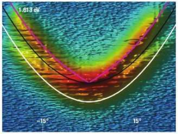
$|\psi(\mathbf{k})|^2 \neq |\psi(-\mathbf{k})|^2$:
Broken time-reversal symmetry.
[Krizhanovskii *et al*, PRB (2009)]

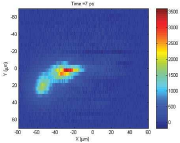
Non-equilibrium superfluidity checklist

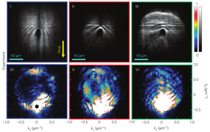
Table 1 | Superfluidity checklist

	Quantized vortices	Landau critical velocity	Metastable persistent flow	Two-fluid hydro-dynamics	Local thermal equilibrium	Solitary waves
Superfluid ^{4}He/cold atom Bose–Einstein condensate	✓	✓	✓	✓	✓	✓
Non-interacting Bose–Einstein condensate	✓	X	X	X	✓	X
Classical irrotational fluid	X	✓	X	✓	✓	✓
Incoherently pumped polariton condensates	✓	X	?	?	X	?
Parametrically pumped polariton condensates	✓	✓	?	?	X	✓









Lagoudakis *et al* Nature Phys. 4, 706 (2008). Utsunomiya *et al* Nature Phys. 4 700 (2008). Amo *et al* Nature 457 291 (2009); Nature Phys (2009)

1. Non-equilibrium model — coherence and strong coupling
   - Green's functions and stability
2. Polarisation and non-equilibrium pattern formation
   - Synchronisation–desynchronisation transition
   - Consequences for steady vortex lattices
3. Condensed spectrum and superfluidity
   - Current-current response and superfluid density

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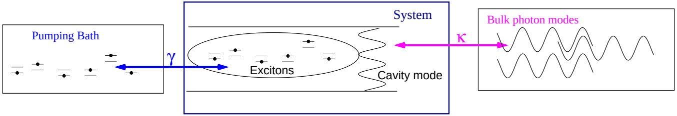
Non-equilibrium approach: Steady state, and fluctuations

$$H = H_{\mathrm{sys}} + H_{\mathrm{sys,bath}} + H_{\mathrm{bath}},$$

$$H_{\mathrm{sys}} = \sum_k \omega_k \psi_k \psi_k^\dagger + \sum_\alpha g_\alpha (\phi_\alpha^\dagger \psi_k + \mathrm{H.c.}) + H_{\mathrm{ex}}[\phi_\alpha, \phi_\alpha^\dagger]$$



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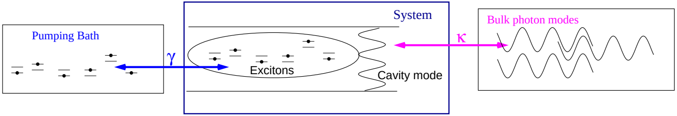

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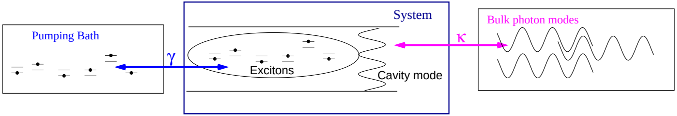

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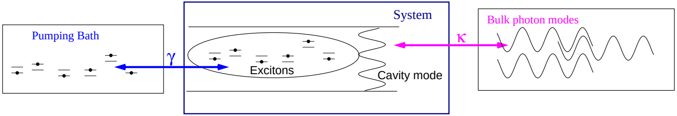

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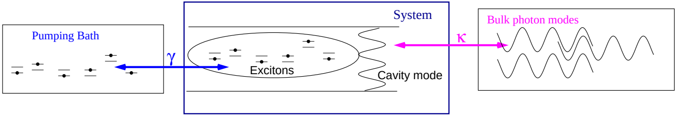

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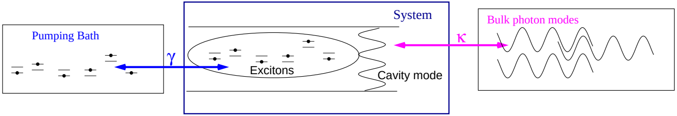

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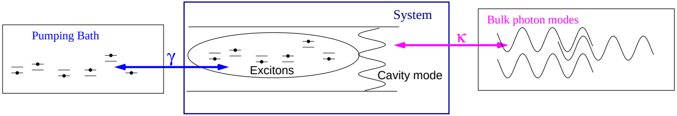

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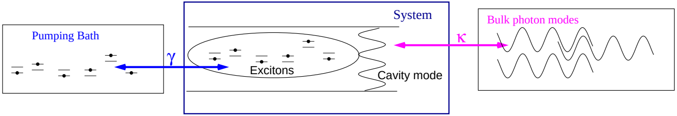

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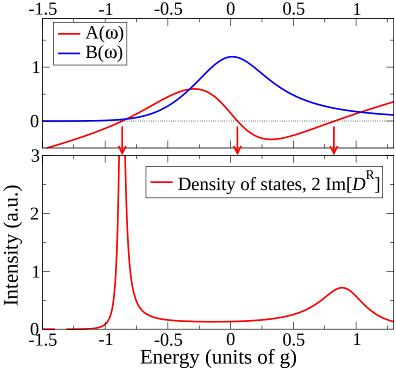
Poles of Green's function and stability

$$\left[D^R(\omega)\right]^{-1} = (\omega - \omega_k^*) + i\alpha(\omega - \mu_{\text{eff}})$$



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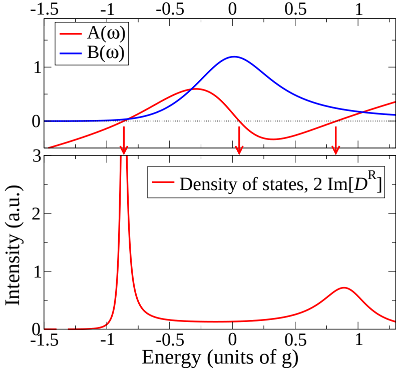

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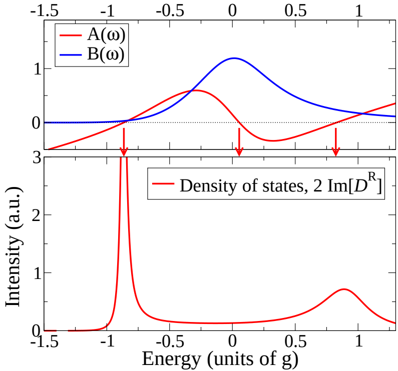

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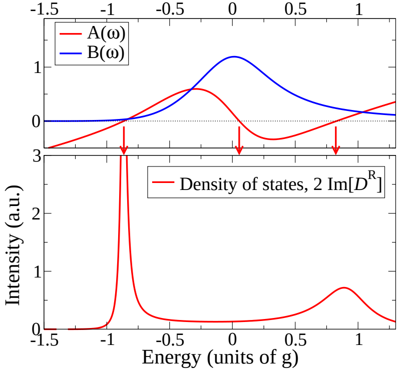

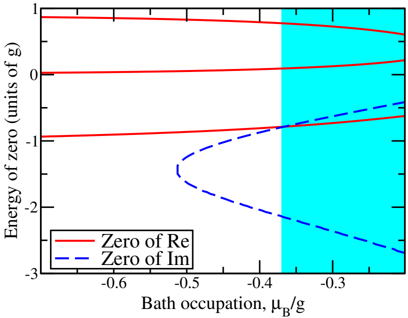

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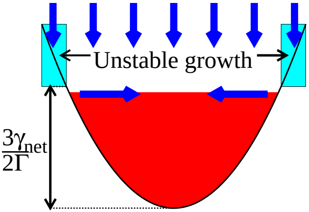
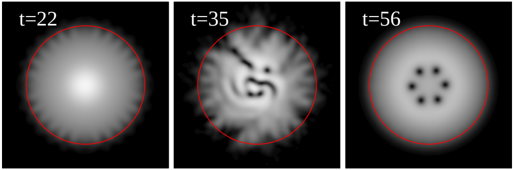
Gross-Pitaevskii equation: Harmonic trap

$$i\partial_t\psi = \left[-\frac{\nabla^2}{2m} + \frac{m\omega^2}{2}r^2 + U|\psi|^2 + i\left(\gamma_{\text{eff}} - \kappa - \Gamma|\psi|^2\right)\right]\psi$$



Pattern formation: Vortex lattices



Non-equilibrium spinor condensate

$$H = H_0[\psi_L] + H_0[\psi_R] + (U_0 - 2U_1)|\psi_L|^2|\psi_R|^2 + \frac{\Delta_\perp}{2}\left(|\psi_L|^2 - |\psi_R|^2\right) + \Delta_\parallel(\psi_L^\dagger \psi_R + \text{H.c.})$$

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- Left-right coupling: U_1
- Magnetic field: $\Delta_\perp$,
- Cross-spin loss terms Γ_1
- Energy-dependent gain η [Wouters *et al* PRB '10]

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Non-equilibrium spinor system: two-mode model

Two-mode case (neglect spatial variation) [Wouters PRB '08]

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Write:

$$\psi_L = \sqrt{R+z}e^{i\phi+i\theta/2},$$
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Simple case $\Gamma_1 = \eta = 0$

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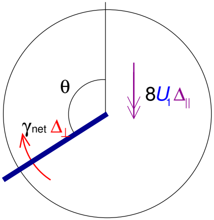

Simple case $\Gamma_1 = \eta = 0$
Josephson regime: $\Delta_\parallel \ll U_1 R$, $z \ll R$.

Damped, driven pendulum
$$\ddot{\theta} + 2\gamma_{\text{net}}\dot{\theta} = 8U_1\Delta_\parallel\frac{\gamma_{\text{net}}}{\Gamma_0}\sin(\theta) - 2\gamma_{\text{net}}\Delta_\perp$$

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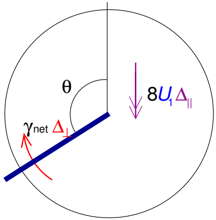

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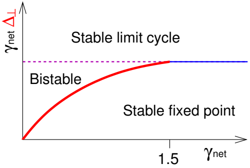

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From two-mode to many mode ($\eta = \Gamma_1 = 0$)

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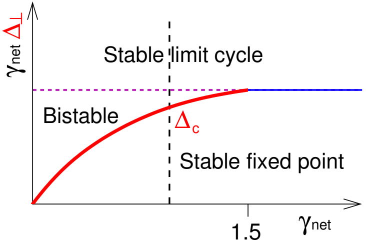
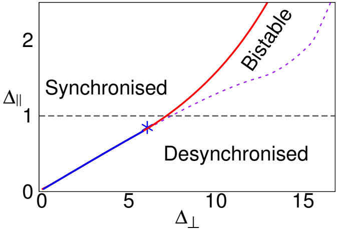


Actual ($\Gamma_1 = \eta = 0$)



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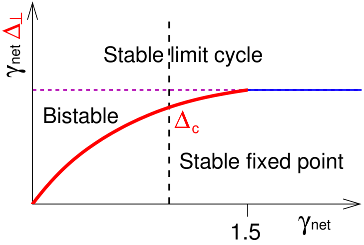

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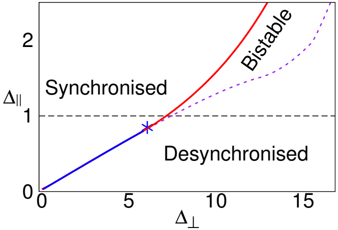
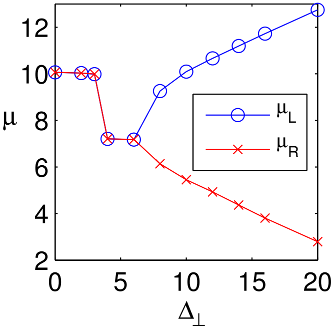
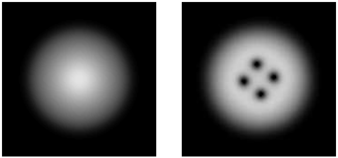


Spatial inhomogeneity

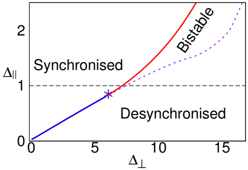




Desynchronised vs circular polarised phase

What happens at large $\Delta_\perp$?

Spinor CGPE $\rightarrow$ Desynchronised.



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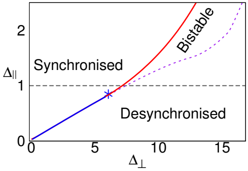
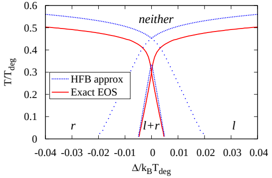


Equilibrium $\rightarrow$ Circular



[Rubo *et al* PLA '06; JK, PRB '08]

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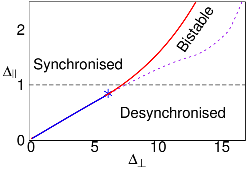

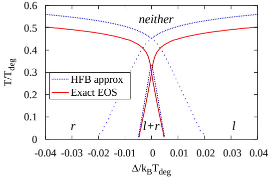

[Rubo *et al* PLA '06; JK, PRB '08]

Energy dependent gain

$$i\partial_t\psi_L = \ldots + i\left(\gamma_{\text{net}} - \Gamma_0|\psi_L|^2 - \Gamma_1|\psi_R|^2 - \eta i\partial_t\right)\psi_L$$

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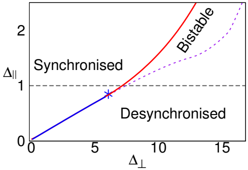

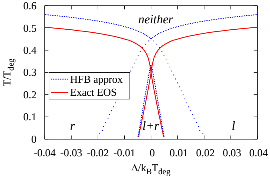

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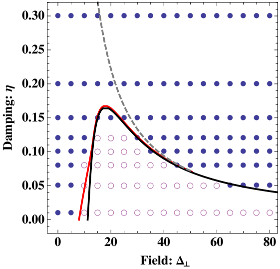

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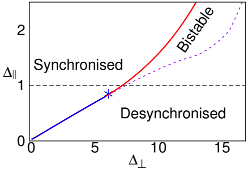

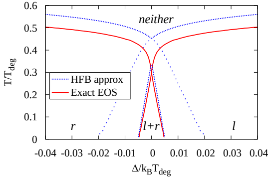

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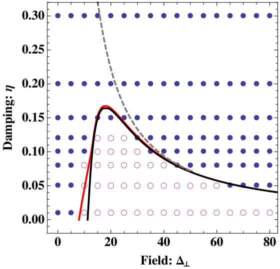
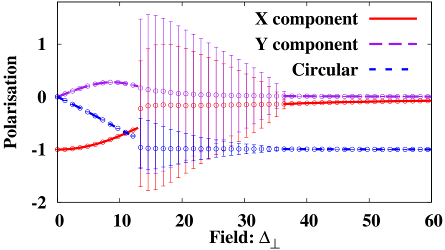
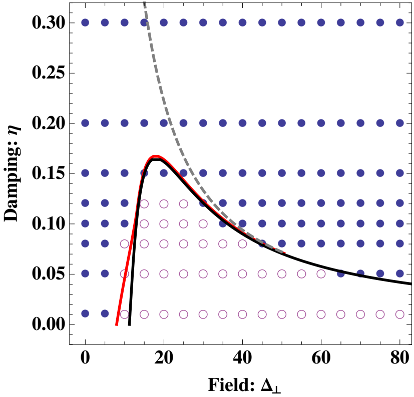




Desynchronisation and pattern formation

Trapping:

Synchronisation unaffected



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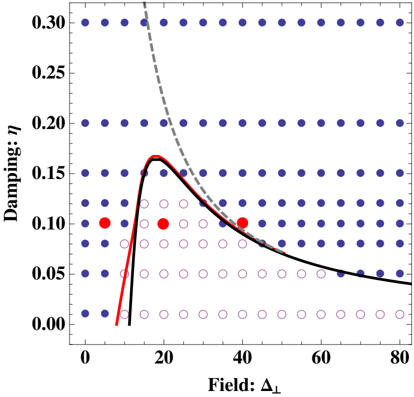
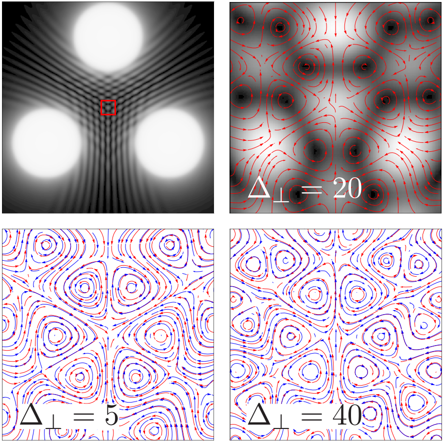


Desynchronisation $\rightarrow$ half-vortex separation:



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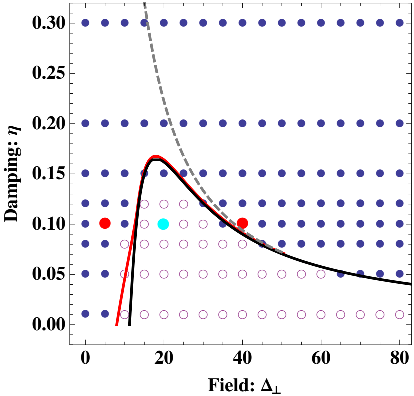

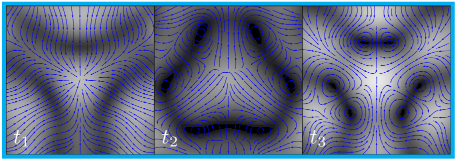

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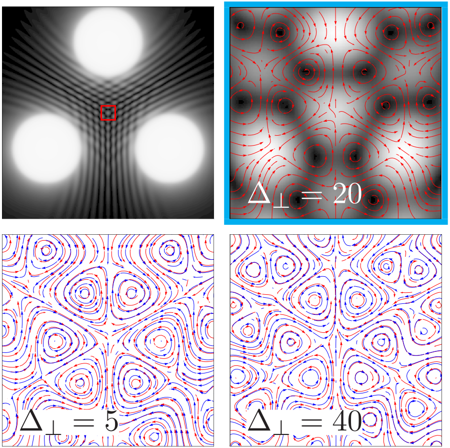


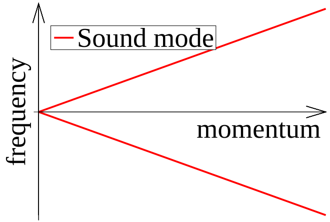
Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega, k)\right]^{-1} = \omega^2 - \xi_k^2$$

With $\xi_k \simeq ck$
Poles:

$$\omega^* = \xi_k$$



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$$\mathrm{Det}\left[D^R(\omega,k)\right]^{-1} = (\omega+i\gamma_{\mathrm{net}})^2+\gamma_{\mathrm{net}}^2-\xi_k^2$$

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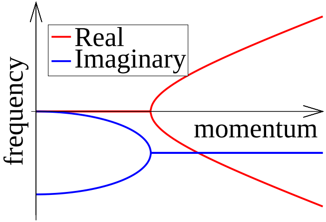

Asking about non-equilibrium superfluidity

Currrent:

$$\mathbf{J} = \rho \mathbf{v} = \Psi^\dagger i \nabla \Psi = |\Psi|^2 \nabla \phi$$

- Response function:

$$\chi_{ij}(\omega = 0, \mathbf{q} \to 0) = \langle [J_i(\mathbf{q}), J_j(-\mathbf{q})] \rangle = \frac{\rho_S}{m} \frac{q_i q_j}{q^2} + \frac{\rho_N}{m} \delta_{ij}$$

- Given D and $J_i = \psi^\dagger(k+q) \frac{2k_i + q_i}{2m} \psi_k$
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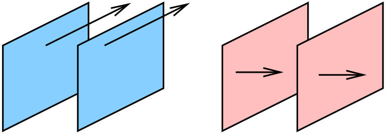

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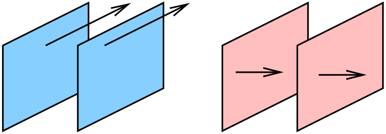

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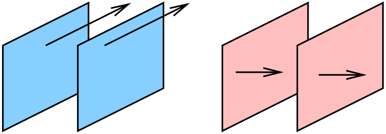

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Non-equilibrium superfluid response

- Superfluid response exists because:

 $$= \frac{i\psi_0 q_i}{2m} (1, -1)\, D^R(q, \omega = 0) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{i\psi_0 q_j}{2m}$$

- $D^R(\omega = 0) \propto 1/\epsilon_q$ despite pumping/decay — superfluid response exists.
- Normal density:

$$\rho_N = \int d^d k \epsilon_k \int \frac{d\omega}{2\pi} \mathrm{Tr}\left[\sigma_z D^K \sigma_z (D^R + D^A)\right]$$

- Is affected by pump/decay;
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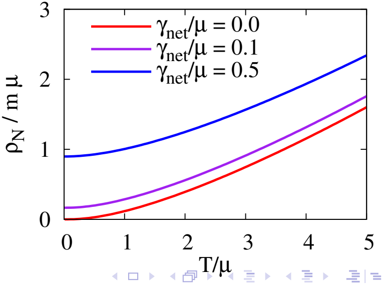


Acknowledgements

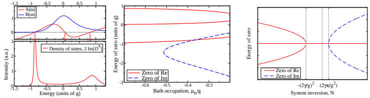
People:

Funding:

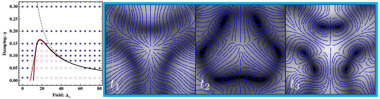
EPSRC
Engineering and Physical Sciences
Research Council

Summary

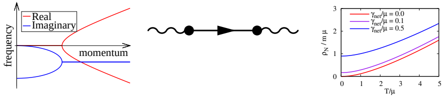
- Non-equilibrium condensation, lasing, and strong-coupling



- Desynchronisation and vortex lattices



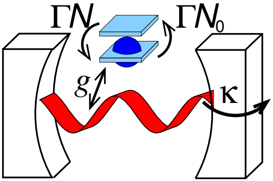
- Survival of superfluid response vs change to spectrum



Extra slides

4. Green's functions
5. Superfluidity
6. Non-equilibrium pattern formation
7. Equilibrium results
8. Spinor problem

Poles and stability for a laser



Maxwell-Bloch equations:

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
$$\partial_t P = -2i\epsilon P - 2\gamma P + g\psi N$$
$$\partial_t N = 2\gamma(N_0 - N) - 2g(\psi^* P + P^* \psi)$$

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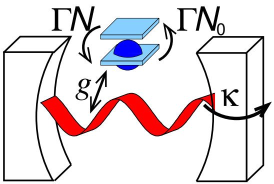

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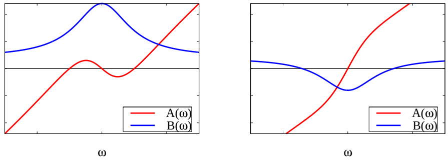

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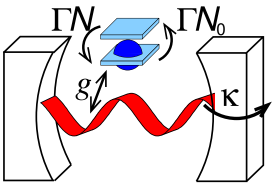

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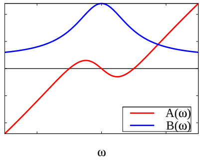

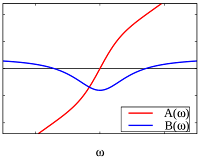

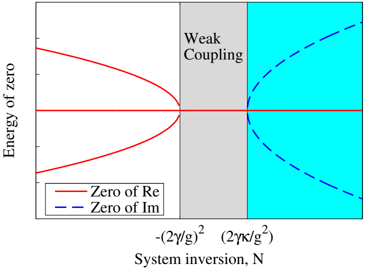

Calculating superfluid response function

- Using Keldysh generating functional

$$\chi_{ij}(q) = -\frac{i}{2}\frac{d^2\mathcal{Z}[f,\theta]}{df_i(q)d\theta_j(-q)}, \qquad \mathcal{Z}[f,\theta] = \int \mathcal{D}\psi \exp(iS[f,\theta])$$

- f, θ couple as force/response current.

$$S[f,\theta] = S + \sum_{k,q} \begin{pmatrix} \bar{\psi}_{cl} & \bar{\psi}_q \end{pmatrix}_{k+q} \begin{pmatrix} \theta_i & f_i + \theta_i \\ f_i - \theta_i & -\theta_i \end{pmatrix}_q \frac{2k_i + q_i}{2m} \begin{pmatrix} \psi_{cl} \\ \psi_q \end{pmatrix}_k$$

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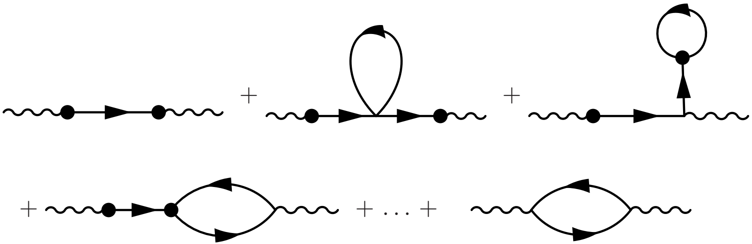

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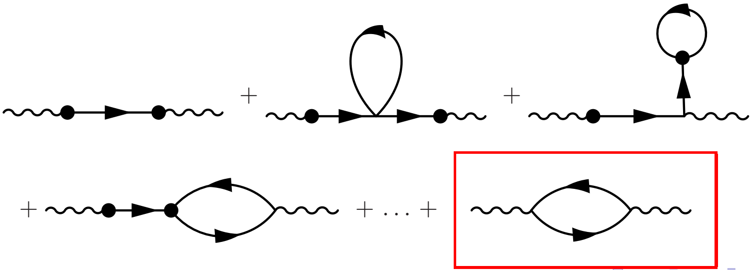

Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit:

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Nonlinear, complex susceptibility $\chi(\psi(r,t))$

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Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit: Gross-Pitaevskii equation

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Gross-Pitaevskii equation: Harmonic trap

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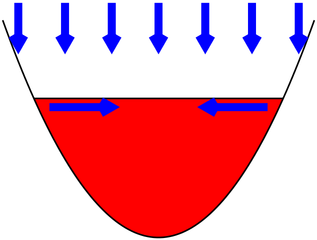

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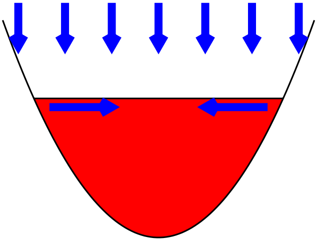
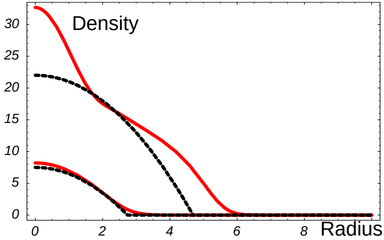
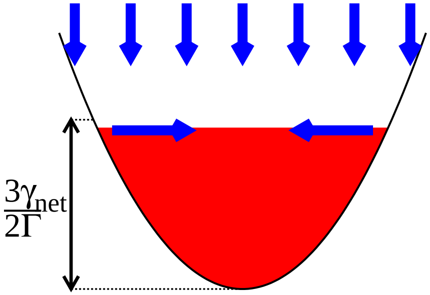




Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t \rho + \nabla\cdot(\rho \mathbf{v}) = (\gamma_{\text{net}} - \Gamma\rho)\rho$$



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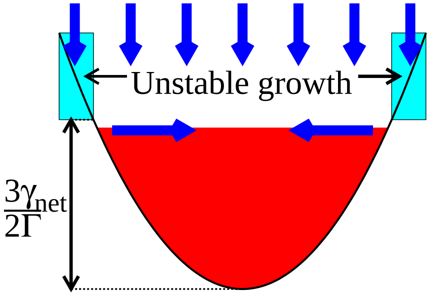

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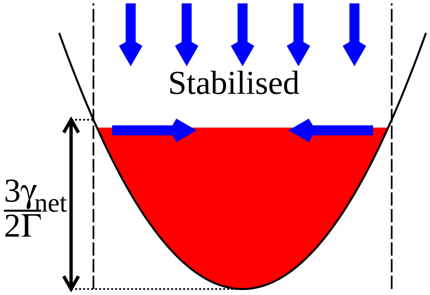

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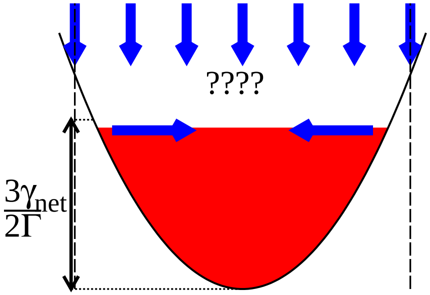
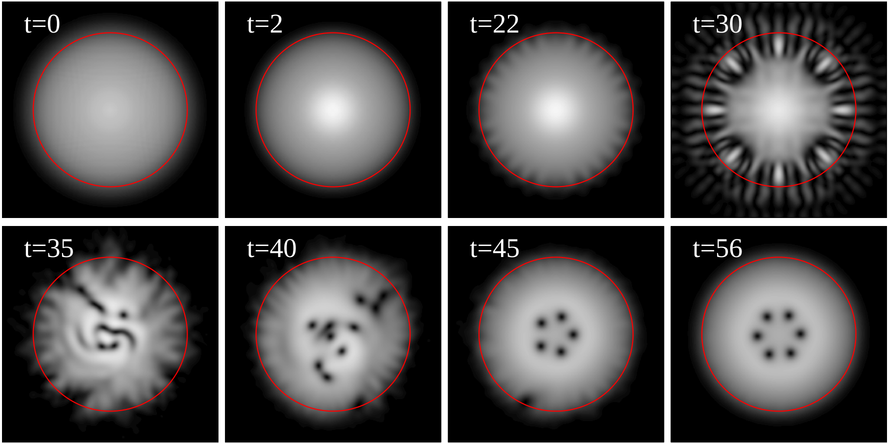
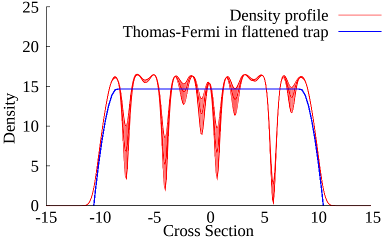
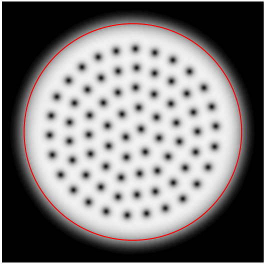


Time evolution:



Why vortices





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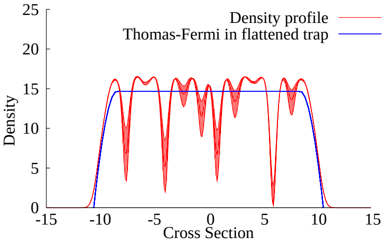

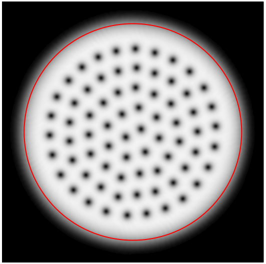

Rotating solution: $i\partial_t\psi = (\mu - 2\Omega L_z)\psi$

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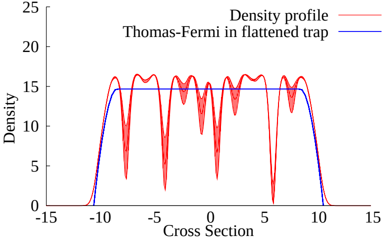

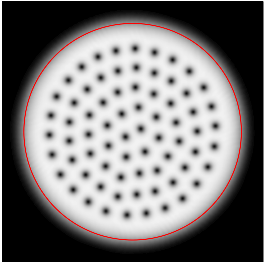

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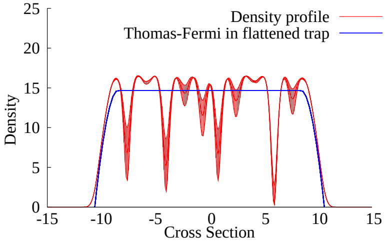

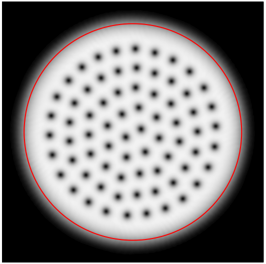

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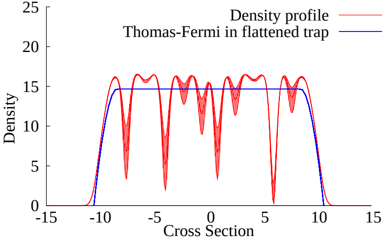

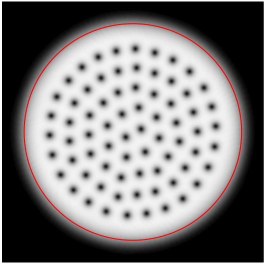

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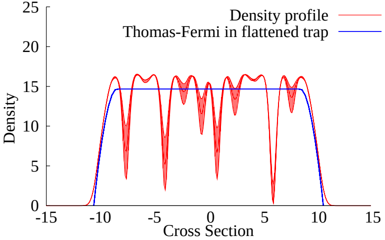

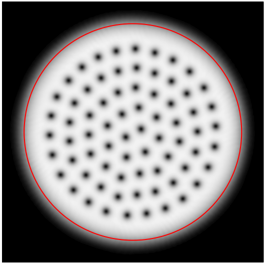

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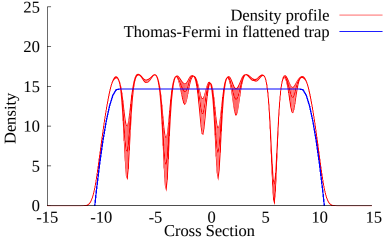

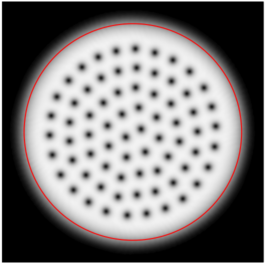

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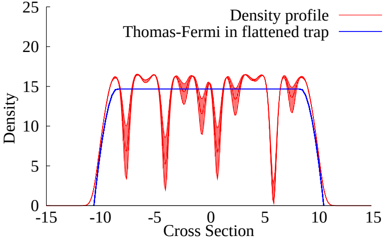

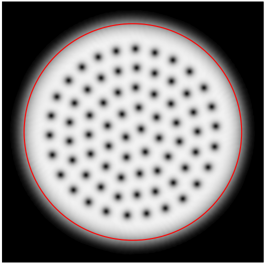

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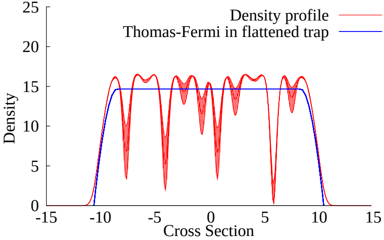

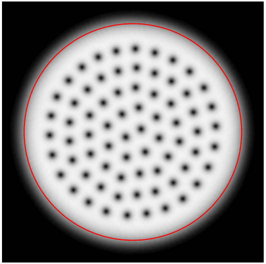

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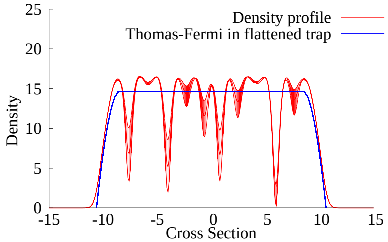

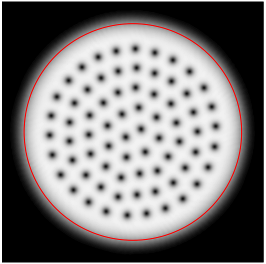

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$$H_{\text{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{A}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \text{H.c.} \right]$$

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Self-consistent polarisation and field

$$\left[- i\partial_t - \omega_0 + \frac{\nabla^2}{2m} \right] \psi = -\frac{1}{\sqrt{A}} \sum_{\alpha} g_{\alpha} P_{\alpha}$$

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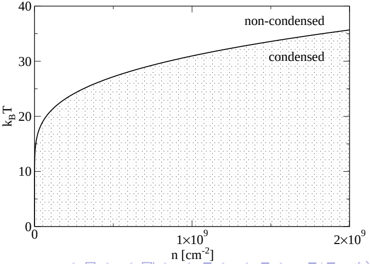

Spin in terms of ~~two~~four-level systems

To include spin, replace 2 level system with 4 levels: $|0\rangle, |L\rangle, |R|\rangle, |LR\rangle$

- Bi-exciton binding $E_{XX} \leftrightarrow U_1$
- Mean-field: find polarisation given ψ_L, ψ_R.
- E_{XX} has weak effect on T_c

[Marchetti *et al* PRB, '08]

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$$\hat{h}_\alpha = \begin{pmatrix} 0 & g\psi_L & g\psi_R & 0 \\ g\psi_L^* & \varepsilon_\alpha - \Delta - \mu & 0 & g\psi_L \\ g\psi_R^* & 0 & \varepsilon_\alpha + \Delta - \mu & g\psi_R \\ 0 & g\psi_L^* & g\psi_R^* & 2(\varepsilon_\alpha - \mu) - E_{XX} \end{pmatrix}$$

- Bi-exciton binding $E_{XX} \leftrightarrow U_1$
- Mean-field: find polarisation given ψ_L, ψ_R.
- E_{XX} has weak effect on T_c

[Marchetti *et al* PRB, '08]

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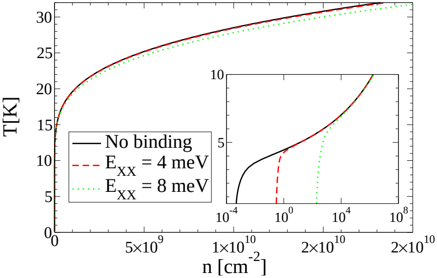

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Non-equilibrium spinor condensate

$$H = H_0[\psi_L] + H_0[\psi_R] + (U_0 - 2U_1)|\psi_L|^2|\psi_R|^2 + \frac{\Delta_\perp}{2}\left(|\psi_L|^2 - |\psi_R|^2\right) + \Delta_\parallel(\psi_L^\dagger\psi_R + \text{H.c.})$$

Spinor Gross-Pitaekvsii equation

$$i\partial_t\psi_L = \left[-\frac{\nabla^2}{2m} + V(r) + U_0|\psi_L|^2 + i\left(\gamma_{\text{eff}} - \kappa - \Gamma_0|\psi_L|^2 \right)\right]\psi_L$$

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Non-equilibrium spinor system: two-mode model

Two-mode case (neglect spatial variation) [Wouters PRB '08]

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Write:

$$\psi_L = \sqrt{R+z}e^{i\phi + i\theta/2},$$
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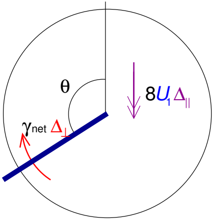

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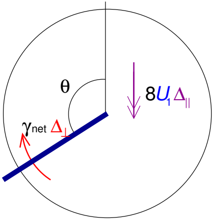

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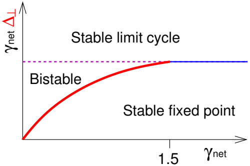
Cartoon:



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