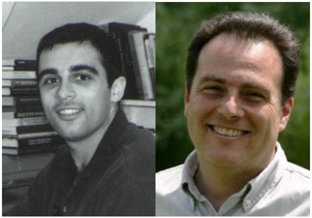
Collective dynamics of Bose–Einstein condensate in optical cavities

J. Keeling, M. J. Bhaseen, B. D. Simons

October 2010

Acknowledgements

People:



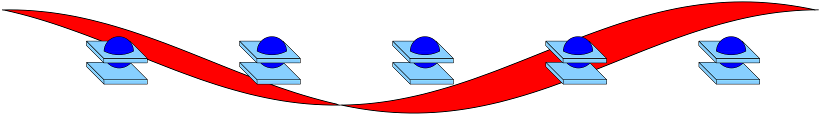
Funding:

EPSRC
Engineering and Physical Sciences
Research Council

Collective emission

Dicke effect: Enhanced emission

$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



- If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$
- Emission: $I \propto \sum_f |\langle f|\psi_k^\dagger S^-|i\rangle|^2$
- For: $|S| = N/2, S^z = M$ get $I \propto \frac{N}{2}\left(\frac{N}{2}+1\right) - M(M-1)$

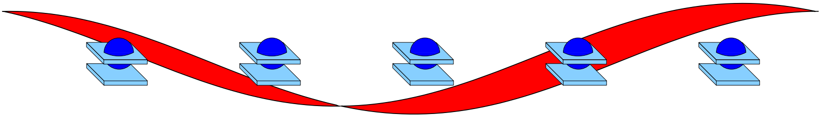
  For $|M| \ll N/2$, rate $\propto N^2$.

[Dicke, Phys. Rev. 1954]

Collective emission

Dicke effect: Enhanced emission

$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



- If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$
- Emission: $I \propto \sum_f |\langle f|\psi_k^\dagger S^-|i\rangle|^2$
- For: $|S| = N/2, S^z = M$ get $I \propto \frac{N}{2}\left(\frac{N}{2}+1\right) - M(M-1)$

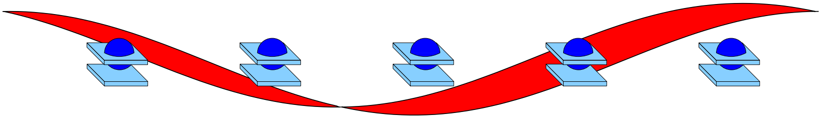
  For $|M| \ll N/2$, rate $\propto N^2$.

[Dicke, Phys. Rev. 1954]

Collective emission

Dicke effect: Enhanced emission

$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



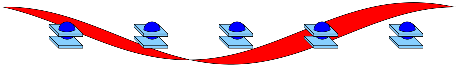
- If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$
- Emission: $I \propto \sum_f |\langle f|\psi_k^\dagger S^-|i\rangle|^2$
- For: $|S| = N/2, S^z = M$ get $I \propto \frac{N}{2}\left(\frac{N}{2}+1\right) - M(M-1)$

  For $|M| \ll N/2$, rate $\propto N^2$.

[Dicke, Phys. Rev. 1954]

Dicke effect and superradiance without a cavity

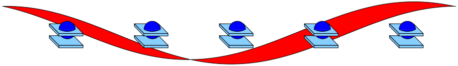
$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



[Dicke, Phys. Rev. 1954]

Dicke effect and superradiance without a cavity

$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



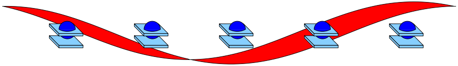
If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$. Many modes ψ_k — integrate out:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+ S^- \rho - S^- \rho S^+ + \rho S^+ S^- \right]$$

[Dicke, Phys. Rev. 1954]

Dicke effect and superradiance without a cavity

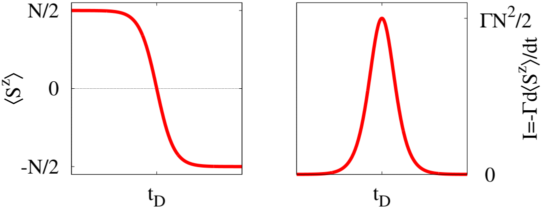
$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$. Many modes ψ_k — integrate out:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+S^-\rho - S^-\rho S^+ + \rho S^+ S^-\right]$$

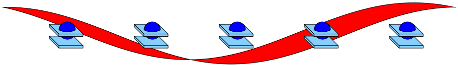
If $S^z = |S| = N/2$ initially: $I \propto -\Gamma \frac{d\langle S^z\rangle}{dt} = \frac{\Gamma N^2}{4} \text{sech}^2\left[\frac{\Gamma N}{2} t\right]$



[Dicke, Phys. Rev. 1954]

Dicke effect and superradiance without a cavity

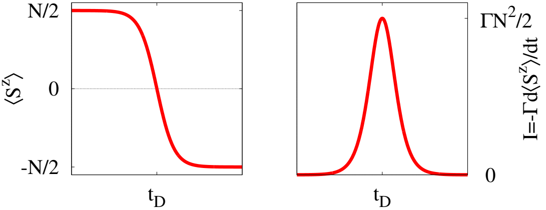
$$H_{\text{int}} = \sum_{k,i} g_k \left(\psi_k^\dagger S_i^- e^{-i\mathbf{k}\cdot\mathbf{r}_i} + \text{H.c.} \right)$$



If $|\mathbf{r}_i - \mathbf{r}_j| \ll \lambda$, use $\sum_i \mathbf{S}_i \to \mathbf{S}$. Many modes ψ_k — integrate out:

$$\frac{d\rho}{dt} = -\frac{\Gamma}{2}\left[S^+S^-\rho - S^-\rho S^+ + \rho S^+ S^-\right]$$

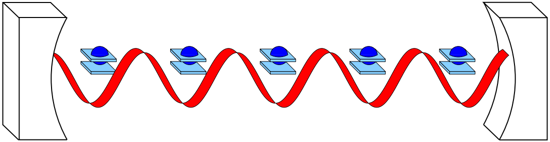
If $S^z = |S| = N/2$ initially: $I \propto -\Gamma \dfrac{d\langle S^z \rangle}{dt} = \dfrac{\Gamma N^2}{4} \text{sech}^2 \left[\dfrac{\Gamma N}{2} t\right]$



[Dicke, Phys. Rev. 1954]

Problem: dipole-dipole interactions dephase.

Collective radiation with a cavity: Dynamics

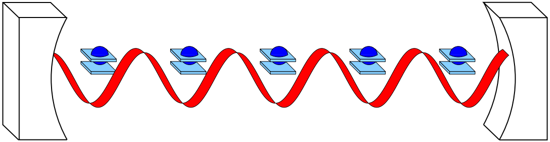


$$H_{\rm int} = \sum_i \left(\psi^\dagger S_i^- + \psi S_i^+ \right)$$

Single cavity mode: oscillations

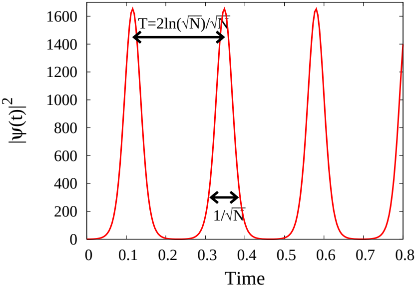
[Bonifacio and Preparata PRA 1970; **JK** PRA 2009]

Collective radiation with a cavity: Dynamics



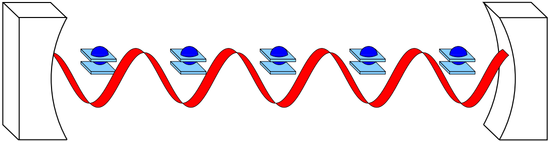
$$H_{\text{int}} = \sum_i \left(\psi^\dagger S_i^- + \psi S_i^+ \right)$$

Single cavity mode: oscillations
If $S^z = |S| = N/2$ initially:



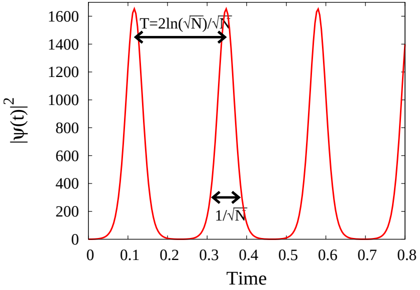
[Bonifacio and Preparata PRA 1970; **JK** PRA 2009]

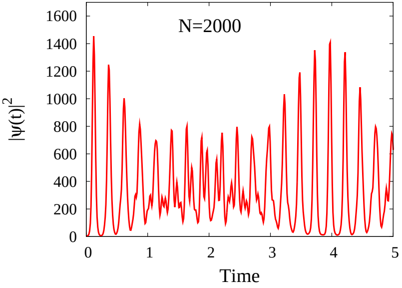
Collective radiation with a cavity: Dynamics



$$H_{\text{int}} = \sum_i \left(\psi^\dagger S_i^- + \psi S_i^+ \right)$$

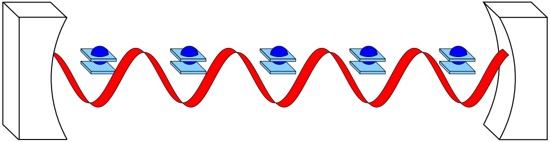
Single cavity mode: oscillations
If $S^z = |S| = N/2$ initially:





[Bonifacio and Preparata PRA 1970; **JK** PRA 2009]

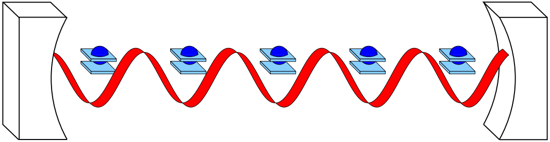
With a cavity: Superradiance phase transition



With detuning: $H = \omega_0 S^z + \omega \psi^\dagger \psi + g\left(\psi^\dagger S^- + \psi S^+\right)$

[Hepp, Lieb, Ann. Phys. 1973]

With a cavity: Superradiance phase transition



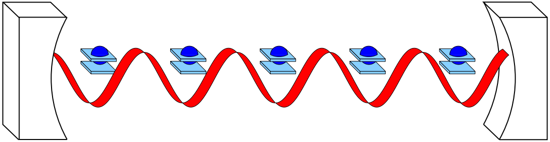
With detuning: $H = \omega_0 S^z + \omega \psi^\dagger \psi + g \left(\psi^\dagger S^- + \psi S^+ \right)$

Mean-field wf: $|\Psi\rangle \to e^{\lambda \psi^\dagger + \eta S^+} |\Omega\rangle$

Spontaneous polarisation if: $Ng^2 > \omega \omega_0$

[Hepp, Lieb, Ann. Phys. 1973]

With a cavity: Superradiance phase transition



With detuning: $H = \omega_0 S^z + \omega\psi^\dagger\psi + g\left(\psi^\dagger S^- + \psi S^+\right)$

Mean-field wf: $|\Psi\rangle \to e^{\lambda\psi^\dagger + \eta S^+}|\Omega\rangle$

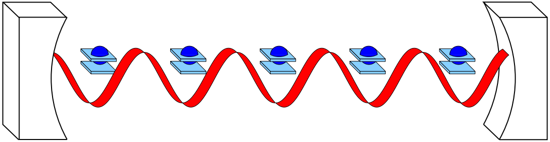
Spontaneous polarisation if: $Ng^2 > \omega\omega_0$

[Hepp, Lieb, Ann. Phys. 1973]

Problem: never occurs. Minimal coupling $(p - eA)^2/2m$

[Rzazewski *et al* Phys. Rev. Lett 1975]

With a cavity: Superradiance phase transition



With detuning: $H = \omega_0 S^z + \omega\psi^\dagger\psi + g\left(\psi^\dagger S^- + \psi S^+\right)$

Mean-field wf: $|\Psi\rangle \to e^{\lambda\psi^\dagger + \eta S^+}|\Omega\rangle$

Spontaneous polarisation if: $Ng^2 > \omega\omega_0$

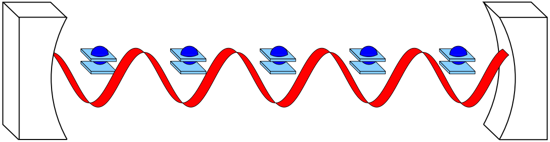
[Hepp, Lieb, Ann. Phys. 1973]

Problem: never occurs. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \quad \Leftrightarrow \quad g(\psi^\dagger S^- + \psi S^+),$$

[Rzazewski *et al* Phys. Rev. Lett 1975]

With a cavity: Superradiance phase transition



With detuning: $H = \omega_0 S^z + \omega\psi^\dagger\psi + g\left(\psi^\dagger S^- + \psi S^+\right)$

Mean-field wf: $|\Psi\rangle \to e^{\lambda\psi^\dagger + \eta S^+}|\Omega\rangle$

Spontaneous polarisation if: $Ng^2 > \omega\omega_0$

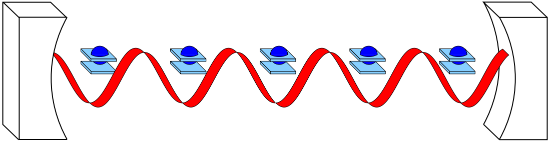
[Hepp, Lieb, Ann. Phys. 1973]

Problem: never occurs. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \quad \Leftrightarrow \quad g(\psi^\dagger S^- + \psi S^+), \qquad \sum_i \frac{A^2}{2m} \quad \Leftrightarrow \quad N\zeta(\psi + \psi^\dagger)^2$$

[Rzazewski *et al* Phys. Rev. Lett 1975]

With a cavity: Superradiance phase transition



With detuning: $H = \omega_0 S^z + \omega\psi^\dagger\psi + g\left(\psi^\dagger S^- + \psi S^+\right)$

Mean-field wf: $|\Psi\rangle \to e^{\lambda\psi^\dagger + \eta S^+}|\Omega\rangle$

Spontaneous polarisation if: $Ng^2 > \omega\omega_0$

[Hepp, Lieb, Ann. Phys. 1973]

Problem: never occurs. Minimal coupling $(p - eA)^2/2m$

$$-\sum_i \frac{e}{m} A \cdot p_i \quad \Leftrightarrow \quad g(\psi^\dagger S^- + \psi S^+), \qquad \sum_i \frac{A^2}{2m} \quad \Leftrightarrow \quad N\zeta(\psi + \psi^\dagger)^2$$

For large N, $\omega \to \omega + 4N\zeta$. Need $Ng^2 > \omega_0(\omega + 4N\zeta)$.
But $g^2/\omega_0 < 4\zeta$. **No transition** [Rzazewski *et al* Phys. Rev. Lett 1975]

Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D} \cdot \mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]

Dicke phase transition: ways out

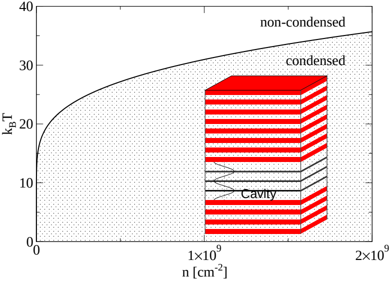
Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D} \cdot \mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]

Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D}\cdot\mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]

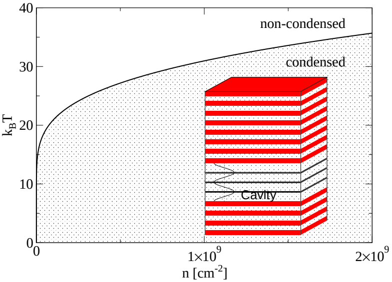


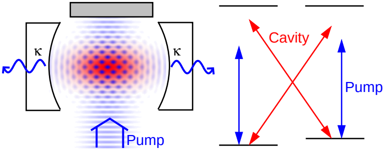
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]

Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D}\cdot\mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]





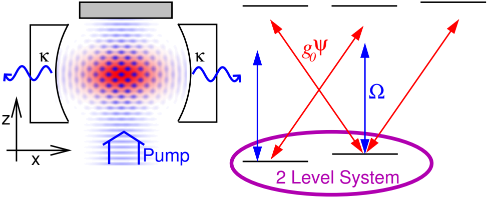
Overview

1. Review Dicke model and superradiance
2. Dynamics of extended Dicke model
   - Fixed points and phase diagram
   - Dynamics and critical slowing down
   - Regions without fixed points
3. Other ways to Dicke superradiance
4. Conclusions

Overview

Extended Dicke model

[Baumann *et al.* Nature 2010]



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:

$\Downarrow$: $|k_x, k_z\rangle = |0, 0\rangle$,

$\Uparrow$: $|k_x, k_z\rangle = |\pm k, \pm k\rangle$,

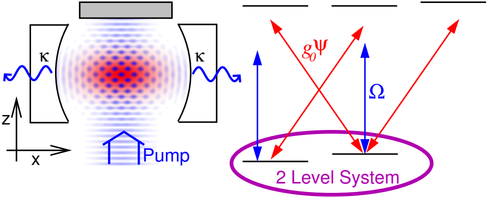
$\omega_0 = 2\omega_{\text{recoil}}$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + U S_z \psi^\dagger\psi$$

N atoms: $|\mathbf{S}| = N/2$

Extended Dicke model

[Baumann *et al.* Nature 2010]



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:

$\Downarrow$: $|k_x, k_z\rangle = |0, 0\rangle$,

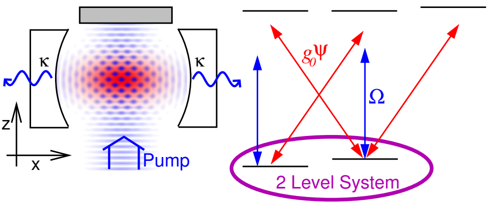
$\Uparrow$: $|k_x, k_z\rangle = |\pm k, \pm k\rangle$,

$\omega_0 = 2\omega_{\text{recoil}}$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-)$$

N atoms: $|\mathbf{S}| = N/2$

Extended Dicke model [Baumann *et al.* Nature 2010]



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:
$\Downarrow$: $|k_x, k_z\rangle = |0, 0\rangle$,
$\Uparrow$: $|k_x, k_z\rangle = |\pm k, \pm k\rangle$,
$\omega_0 = 2\omega_{\text{recoil}}$

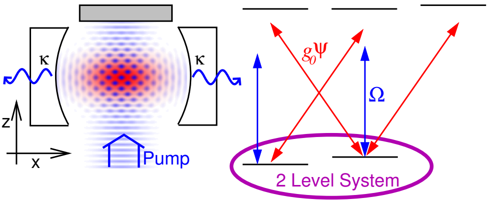
Feedback: $U \propto \dfrac{g_0^2}{\omega_c - \omega_a}$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + U S_z \psi^\dagger \psi.$$

N atoms: $|\mathbf{S}| = N/2$

Extended Dicke model

[Baumann *et al.* Nature 2010]



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:

$\Downarrow$: $\quad |k_x, k_z\rangle = |0, 0\rangle$,

$\Uparrow$: $\quad |k_x, k_z\rangle = |\pm k, \pm k\rangle$,

$\omega_0 = 2\omega_{\text{recoil}}$

Feedback: $U \propto \dfrac{g_0^2}{\omega_c - \omega_a}$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + U S_z \psi^\dagger \psi.$$

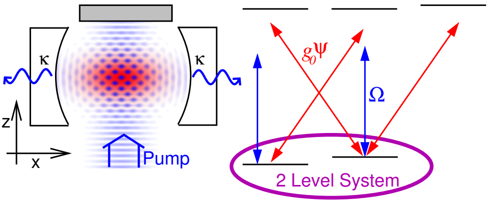
N atoms: $|\mathbf{S}| = N/2$

Add decay:

$$\begin{aligned}
\dot{S}^- &= -i(\omega_0 + U\psi^\dagger\psi)S^- + 2i(g\psi + g'\psi^\dagger)S^z \\
\dot{S}^z &= -ig(\psi S^+ - \psi^\dagger S^-) + ig'(\psi S^- - \psi^\dagger S^+) \\
\dot{\psi} &= -\left[\kappa + i(\omega + US^z)\right]\psi - igS^- - ig'S^+
\end{aligned}$$

Extended Dicke model

[Baumann *et al.* Nature 2010]



2 Level system, $|\Downarrow\rangle, |\Uparrow\rangle$:
$\Downarrow$: $|k_x, k_z\rangle = |0, 0\rangle$,
$\Uparrow$: $|k_x, k_z\rangle = |\pm k, \pm k\rangle$,
$\omega_0 = 2\omega_{\text{recoil}}$

Feedback: $U \propto \dfrac{g_0^2}{\omega_c - \omega_a}$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-) + U S_z \psi^\dagger\psi.$$

N atoms: $|\mathbf{S}| = N/2$

Add decay:

$$\begin{aligned}
\dot{S}^- &= -i(\omega_0 + U\psi^\dagger\psi)S^- + 2i(g\psi + g'\psi^\dagger)S^z \\
\dot{S}^z &= -ig(\psi S^+ - \psi^\dagger S^-) + ig'(\psi S^- - \psi^\dagger S^+) \\
\dot{\psi} &= -\left[\kappa + i(\omega + US^z)\right]\psi - igS^- - ig'S^+
\end{aligned}$$

NB. $\omega, \kappa, g\sqrt{N} \sim$ MHz, $\omega_0 \sim$ kHz. Much slower decay.

Fixed points at $U = 0$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-)$$

Fixed points $\dot{\mathbf{S}}, \dot{\psi} = 0$.

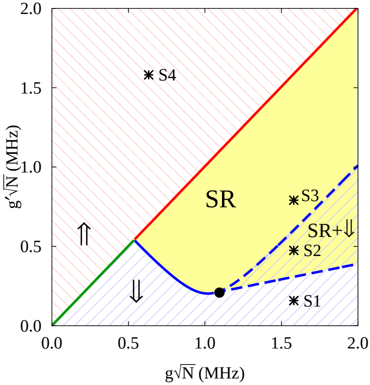
- $S^z = \pm N/2, \psi = 0$ always present
- $\psi \neq 0$ if g, g' large.

Fixed points at $U = 0$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-)$$

Fixed points $\dot{\mathbf{S}}, \dot{\psi} = 0$.

- $S^z = \pm N/2, \psi = 0$ always present
- $\psi \neq 0$ if g, g' large.

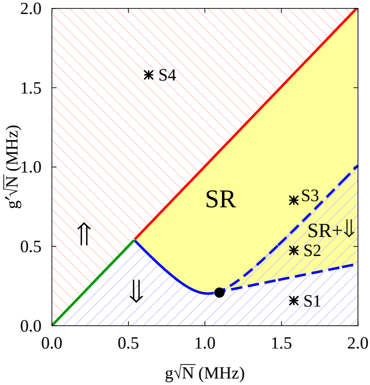


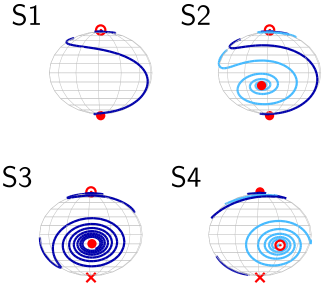
Fixed points at $U = 0$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger S^- + \psi S^+) + g'(\psi^\dagger S^+ + \psi S^-)$$

Fixed points $\dot{\mathbf{S}}, \dot{\psi} = 0$.

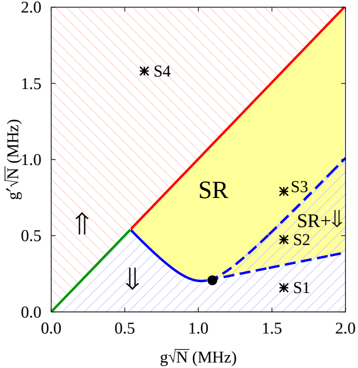
- $S^z = \pm N/2, \psi = 0$ always present
- $\psi \neq 0$ if g, g' large.





Boundaries $U = 0$

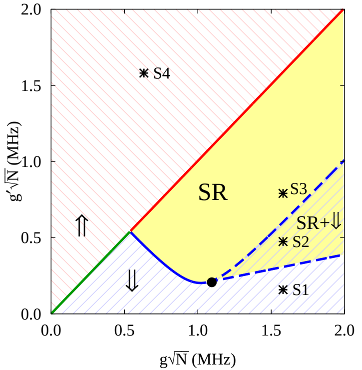
$\kappa \neq 0$



Boundaries $U = 0$

$\kappa \neq 0$

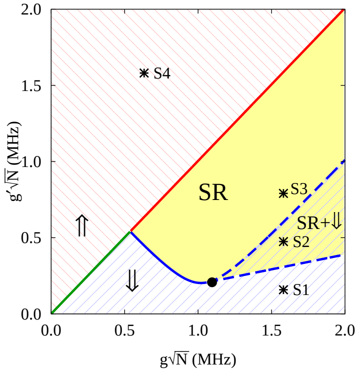
(red line), (green line) $\frac{g'}{g} = \sqrt{\frac{(\omega+\omega_0)^2+\kappa^2}{(\omega-\omega_0)^2+\kappa^2}}$



Boundaries $U = 0$

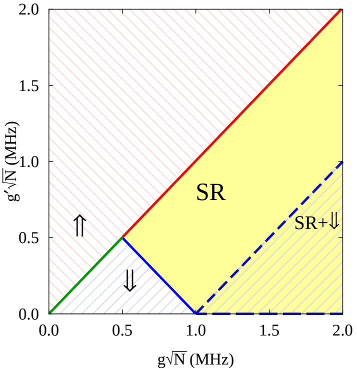
$\kappa \neq 0$

—, — $\dfrac{g'}{g} = \sqrt{\dfrac{(\omega+\omega_0)^2+\kappa^2}{(\omega-\omega_0)^2+\kappa^2}}$

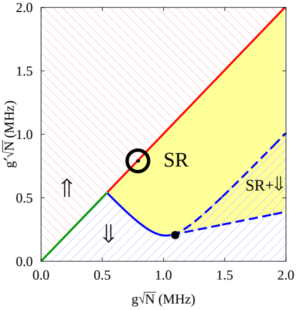


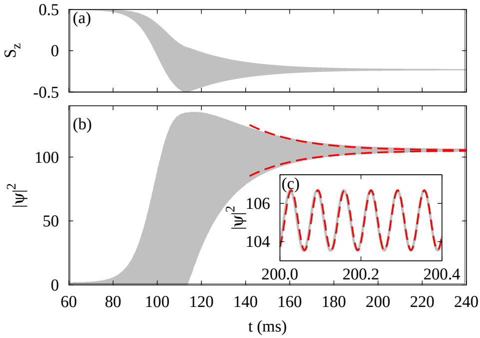
$\kappa = 0$:

—$N(g+g')^2 = \omega\omega_0$



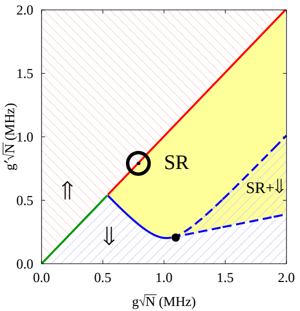
Slow dynamics near critical g'/g

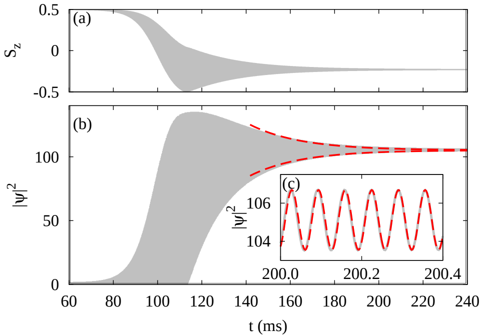




$\omega, \kappa, g\sqrt{N} \sim$ MHz, $\omega_0 \sim$ kHz. Much slower decay.

Slow dynamics near critical g'/g

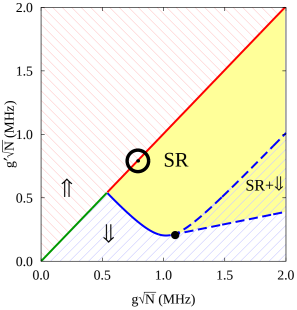


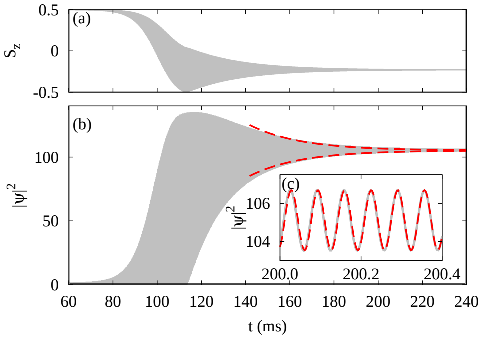


$\omega, \kappa, g\sqrt{N} \sim$ MHz, $\omega_0 \sim$ kHz. Much slower decay.

- Treating ω_0/κ perturbatively, linear stability gives $\mathrm{Im}(\nu) = -\dfrac{\kappa\omega_0^2}{\kappa^2 + \omega^2}$
- For large κ/ω_0, adiabatically eliminate ψ:
  $\partial_t \mathbf{S} = \{\mathbf{S}, H\} - \Gamma \mathbf{S} \times (\mathbf{S} \times \hat{z})$, $\quad H = \omega_0 S_z - \Lambda_+ S_x^2 - \Lambda_- S_y^2$
  $\Lambda_\pm = \frac{\omega}{\kappa^2+\omega^2}(g \pm g')^2$, $\Gamma = \frac{2\kappa}{\kappa^2+\omega^2}(g'^2 - g^2)$

Slow dynamics near critical g'/g





$\omega, \kappa, g\sqrt{N} \sim$ MHz, $\omega_0 \sim$ kHz. Much slower decay.

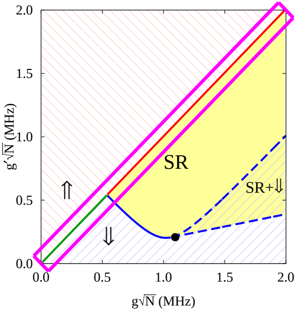
- Treating ω_0/κ perturbatively, linear stability gives $\mathrm{Im}(\nu) = -\dfrac{\kappa\omega_0^2}{\kappa^2+\omega^2}$
- For large κ/ω_0, adiabatically eliminate ψ:
  $\partial_t \mathbf{S} = \{\mathbf{S}, H\} - \Gamma \mathbf{S} \times (\mathbf{S} \times \hat{z}), \quad H = \omega_0 S_z - \Lambda_+ S_x^2 - \Lambda_- S_y^2$
  $\Lambda_\pm \equiv \frac{\omega}{\kappa^2+\omega^2}(g \pm g')^2,\ \Gamma \equiv \frac{2\kappa}{\kappa^2+\omega^2}(g'^2 - g^2)$

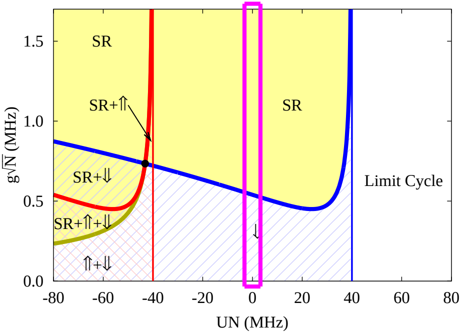
Finite U phase diagram, $g = g'$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger + \psi)(S^+ + S^-) + U S_z \psi^\dagger\psi$$

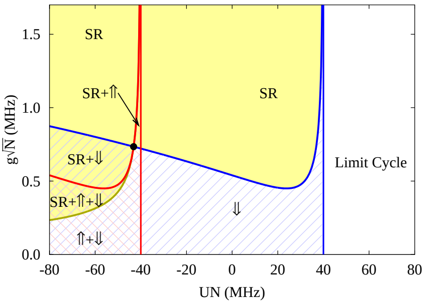
Finite U phase diagram, $g = g'$

$$H = \omega\psi^\dagger\psi + \omega_0 S^z + g(\psi^\dagger + \psi)(S^+ + S^-) + US_z\psi^\dagger\psi$$



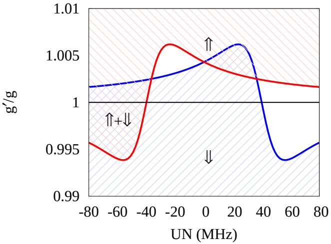


Explaining finite U phase diagram

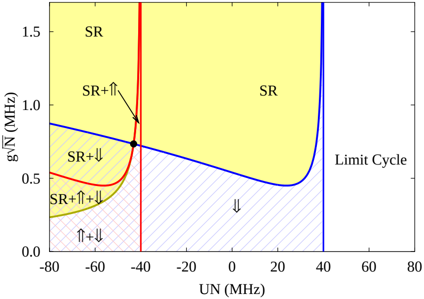


⇑ vv ⇓ instability:

$$\frac{g'}{g} \rightarrow$$

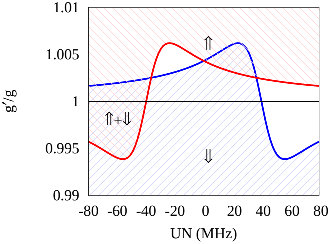


Explaining finite U phase diagram



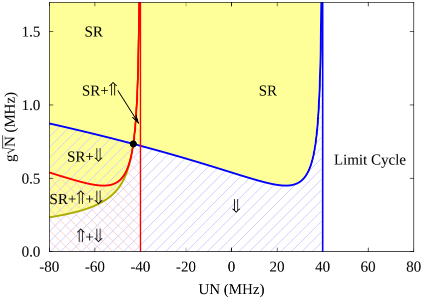
⇑ vv ⇓ instability:

$\frac{g'}{g} \rightarrow$



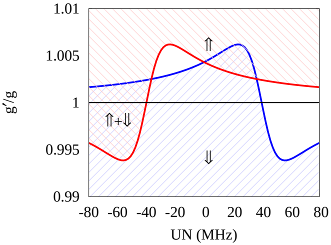
If $g = g'$, analytic $\psi \neq 0$ solution. $\partial_t S^z = ig(\psi + \psi^\dagger)(S^- - S^+)$

Explaining finite U phase diagram



⇑ vv ⇓ instability:

$\frac{g'}{g} \rightarrow$

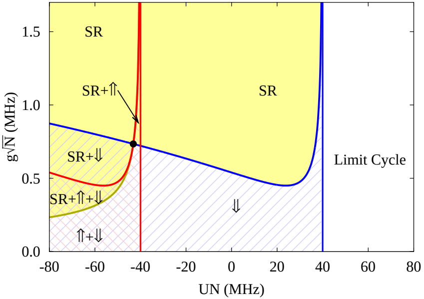


If $g = g'$, analytic $\psi \neq 0$ solution. $\partial_t S^z = ig(\psi + \psi^\dagger)(S^- - S^+)$

- If $|UN| < 2\omega$: $S^\pm = S^x$

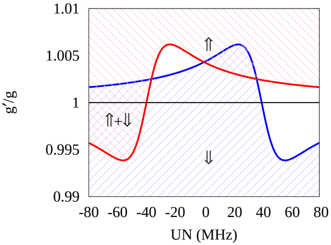
$$S_z = -\frac{\omega}{U} \pm \sqrt{\frac{g^2(4\omega^2 - U^2N^2) - U\omega_0\kappa^2}{U^2(\omega_0 U + 4g^2)}}$$

Explaining finite U phase diagram



⇑ vv ⇓ instability:

$\frac{g'}{g} \rightarrow$



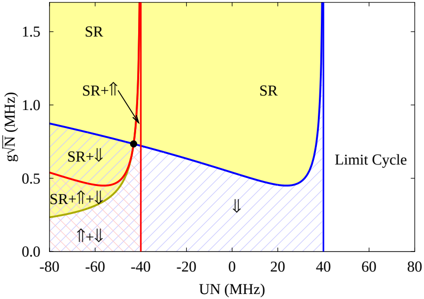
If $g = g'$, analytic $\psi \neq 0$ solution. $\partial_t S^z = ig(\psi + \psi^\dagger)(S^- - S^+)$

- If $|UN| < 2\omega$: $S^\pm = S^x$

$$S_z = -\frac{\omega}{U} \pm \sqrt{\frac{g^2(4\omega^2 - U^2N^2) - U\omega_0\kappa^2}{U^2(\omega_0 U + 4g^2)}}$$

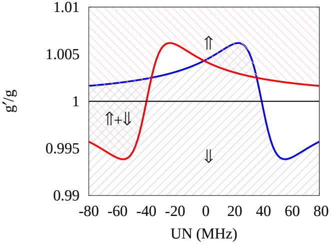
- If $UN < -2\omega$ Alternate SR solution

Explaining finite U phase diagram



⇑ vv ⇓ instability:

$\frac{g'}{g} \rightarrow$



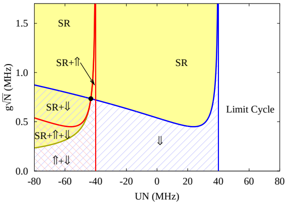
If $g = g'$, analytic $\psi \neq 0$ solution. $\partial_t S^z = ig(\psi + \psi^\dagger)(S^- - S^+)$

- If $|UN| < 2\omega$: $S^\pm = S^x$

$$S_z = -\frac{\omega}{U} \pm \sqrt{\frac{g^2(4\omega^2 - U^2N^2) - U\omega_0\kappa^2}{U^2(\omega_0 U + 4g^2)}}$$

- If $UN < -2\omega$ Alternate SR solution
- If $UN > 2\omega$ **No stable fixed points**

Persistent optomechanical oscillations

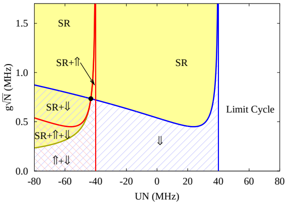


$$\partial_t S^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^\dagger)S^z$$

$$\partial_t S^z = i(\psi + \psi^\dagger)(S^- - S^+)$$

$$\partial_t \psi = -\left[\kappa + i(\omega + US^z)\right]\psi - ig(S^- + S^+)$$

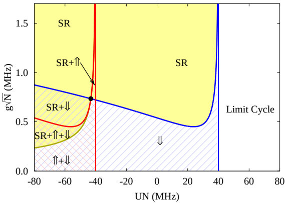
Persistent optomechanical oscillations



$$\partial_t S^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^\dagger)S^z$$
$$\partial_t S^z = i(\psi + \psi^\dagger)(S^- - S^+)$$
$$\partial_t \psi = -\left[\kappa + i(\omega + US^z)\right]\psi - ig(S^- + S^+)$$

Fix $S^z = -\omega/U$ **if** $\mathrm{Re}(\psi) = 0$.

Persistent optomechanical oscillations



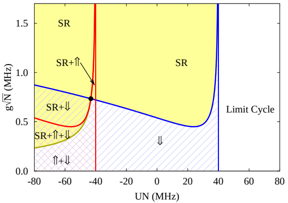
$$\partial_t S^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^\dagger)S^z$$

$$\partial_t S^z = i(\psi + \psi^\dagger)(S^- - S^+)$$

$$\partial_t \psi = -\left[\kappa + i(\omega + US^z)\right]\psi - ig(S^- + S^+)$$

Fix $S^z = -\omega/U$ **if** $\mathrm{Re}(\psi) = 0$.

Persistent optomechanical oscillations



$$\partial_t S^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^\dagger)S^z$$
$$\partial_t S^z = i(\psi + \psi^\dagger)(S^- - S^+)$$
$$\partial_t \psi = -\left[\kappa + i(\omega + US^z)\right]\psi - ig(S^- + S^+)$$

Fix $S^z = -\omega/U$ **if** $\mathrm{Re}(\psi) = 0$.

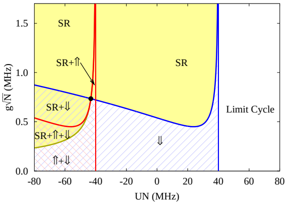
Writing

$$S^- = re^{-i\theta} \quad r = \sqrt{\frac{N^2}{4} - (S^z)^2}$$

Get:

$$\partial_t \theta = \omega_0 + U|\psi|^2$$
$$(\partial_t + \kappa)\psi = -2igr\cos(\theta)$$

Persistent optomechanical oscillations



$$\partial_t S^- = -i(\omega_0 + U|\psi|^2)S^- + 2ig(\psi + \psi^\dagger)S^z$$
$$\partial_t S^z = i(\psi + \psi^\dagger)(S^- - S^+)$$
$$\partial_t \psi = -\left[\kappa + i(\omega + US^z)\right]\psi - ig(S^- + S^+)$$

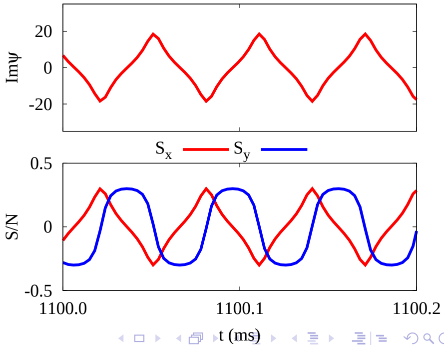
Fix $S^z = -\omega/U$ **if** $\mathrm{Re}(\psi) = 0$.

Writing

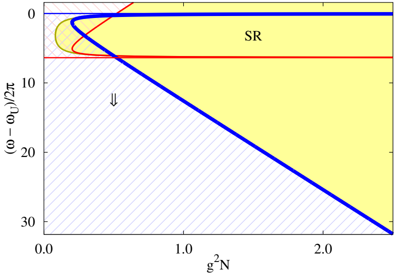
$$S^- = re^{-i\theta} \quad r = \sqrt{\frac{N^2}{4} - (S^z)^2}$$

Get:

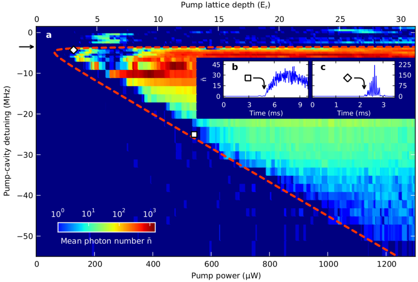
$$\partial_t \theta = \omega_0 + U|\psi|^2$$
$$(\partial_t + \kappa)\psi = -2igr\cos(\theta)$$



Comparison to experiment $UN = -40\text{MHz}$



[**JK** *et al* PRL 2010]



[Baumann *et al* Nature 2010]

Overview

Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D}\cdot\mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]

Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

Two-level systems — dipole-dipole coupling

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + g(S^+ + S^-)(\psi + \psi^\dagger) + N\zeta(\psi + \psi^\dagger)^2 - \eta(S^+ - S^-)^2$$

(nb $g^2, \zeta, \eta \propto 1/V$).

Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

Two-level systems — dipole-dipole coupling

$$H = \omega_0 S^z + \omega \psi^\dagger \psi + g(S^+ + S^-)(\psi + \psi^\dagger) + N\zeta(\psi + \psi^\dagger)^2 - \eta(S^+ - S^-)^2$$

(nb $g^2, \zeta, \eta \propto 1/V$).

Ferroelectric polarisation if $\omega_0 < 2\eta N$

Ferroelectric transition

Atoms in Coulomb gauge

$$H = \sum \omega_k a_k^\dagger a_k + \sum_i [p_i - eA(r_i)]^2 + V_{\text{coul}}$$

Two-level systems — dipole-dipole coupling

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + g(S^+ + S^-)(\psi + \psi^\dagger) + N\zeta(\psi + \psi^\dagger)^2 - \eta(S^+ - S^-)^2$$

(nb $g^2, \zeta, \eta \propto 1/V$).

Ferroelectric polarisation if $\omega_0 < 2\eta N$

Gauge transform to dipole gauge $\mathbf{D} \cdot \mathbf{r}$

$$H = \omega_0 S^z + \omega\psi^\dagger\psi + \bar{g}(S^+ - S^-)(\psi - \psi^\dagger)$$

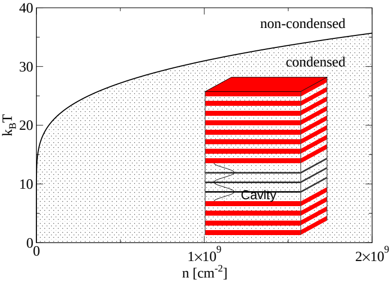
"Dicke" transition at $\omega_0 < N\bar{g}^2/\omega \equiv 2\eta N$

But, ψ describes **electric displacement**

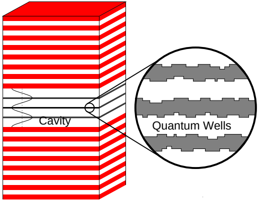
Dicke phase transition: ways out

Problem: $g^2/\omega_0 < 4\zeta$ for intrinsic parameters.
Solutions:

- **Non-solution** Ferroelectric transition in $\mathbf{D} \cdot \mathbf{r}$ gauge. [**JK** JPCM 2007]
- Grand canonical ensemble:
  - If $H \to H - \mu(S^z + \psi^\dagger\psi)$, need only: $g^2 N > (\omega - \mu)(\omega_0 - \mu)$
  - Incoherent pumping — polaritons. [**JK** Semicond. Sci. Technol. 2007]
- Dissociate g, ω_0, e.g. Raman Scheme: $\omega_0 \ll \omega$. [Dimer *et al* PRA 2007; Baumann *et al* Nature 2010]



Chemical potential and Dicke model

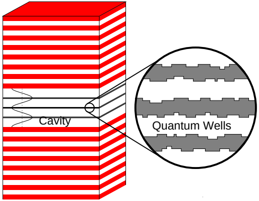


$$H = \omega\psi^\dagger\psi + \sum_i \epsilon_i S_i^z + g_i(\psi S_i^+ + \text{H.c.})$$

Schematic equation:

$$\omega - \mu = \frac{g^2 N}{\omega_0 - \mu} \to \sum_i \frac{g_i^2}{\epsilon_i - \mu} \tanh\left[\beta \frac{1}{2}(\epsilon_i - \mu)\right]$$

Chemical potential and Dicke model

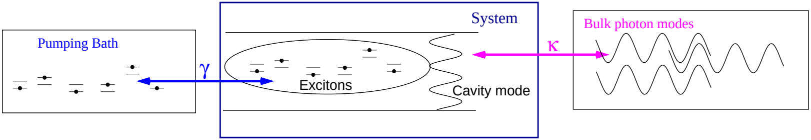


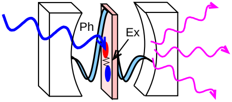
$$H = \omega\psi^\dagger\psi + \sum_i \epsilon_i S_i^z + g_i(\psi S_i^+ + \text{H.c.})$$

Schematic equation:

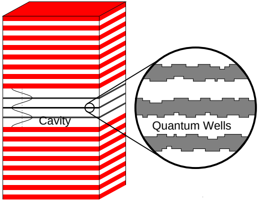
$$\omega - \mu = \frac{g^2 N}{\omega_0 - \mu} \to \sum_i \frac{g_i^2}{\epsilon_i - \mu} \tanh\left[\beta\frac{1}{2}(\epsilon_i - \mu)\right]$$

Open system





Chemical potential and Dicke model

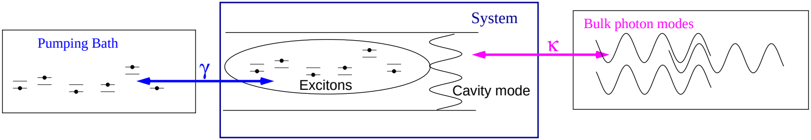


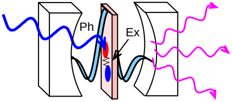
$$H = \omega\psi^\dagger\psi + \sum_i \epsilon_i S_i^z + g_i(\psi S_i^+ + \text{H.c.})$$

Schematic equation:

$$\omega - \mu = \frac{g^2 N}{\omega_0 - \mu} \to \sum_i \frac{g_i^2}{\epsilon_i - \mu} \tanh\left[\beta\frac{1}{2}(\epsilon_i - \mu)\right]$$

Open system



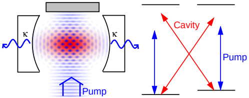


$$\psi(t) = \psi e^{-i\mu_S t}$$

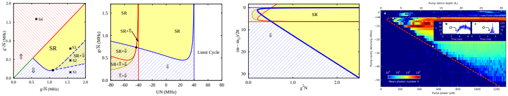
$$(\omega - \mu_S - i\kappa)\psi = -\sum_i g_i\langle S_i^-\rangle = -\sum_i \chi(\psi_0, \mu_S)\psi$$

Summary

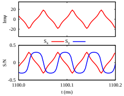
- Realisation of (modified) superradiance transition



- For $g \neq g'$, $U \neq 0$, wide variety of phases



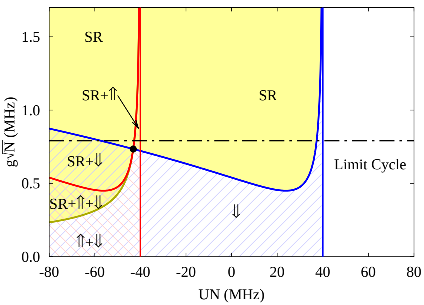
- Slow dynamics and collective oscillations



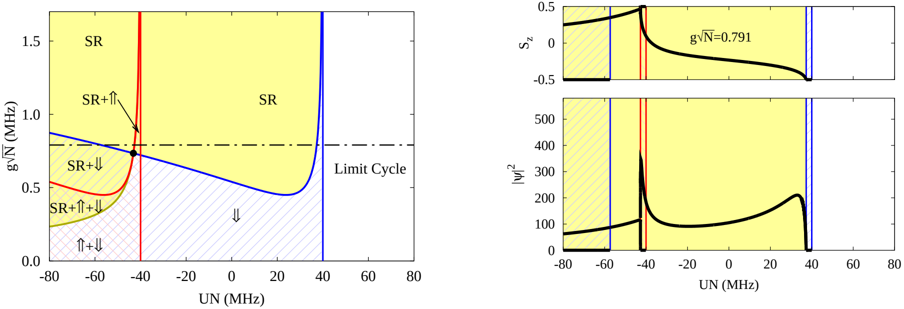
Extra slides

5. Numerical confirmation of FP
6. Dicke Oscillations
7. Extensions to atomic Dicke realisation

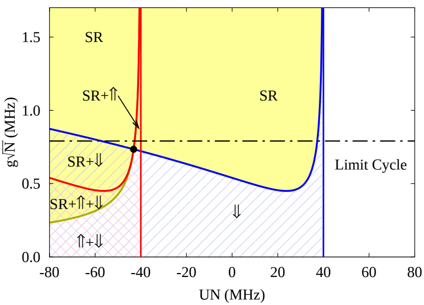
Numerical confirmation of fixed points

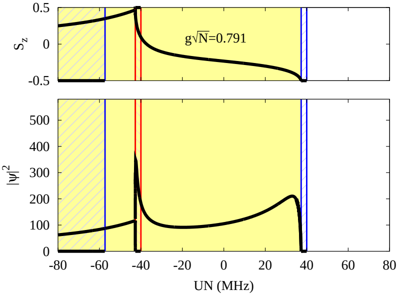


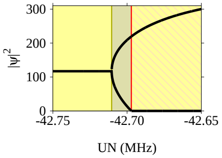
Numerical confirmation of fixed points



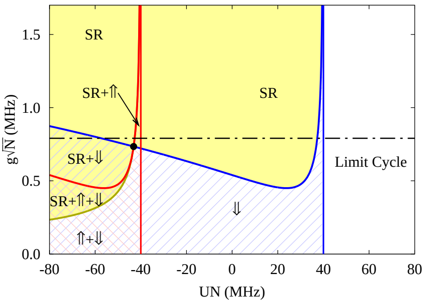
Numerical confirmation of fixed points



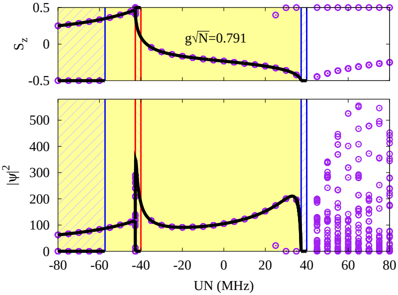


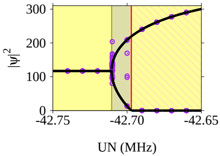


Numerical confirmation of fixed points



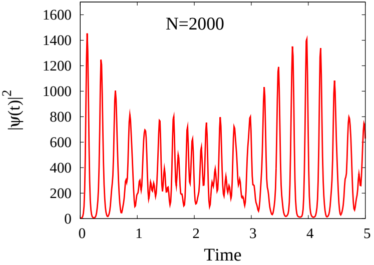
$T = 360\text{ms}$





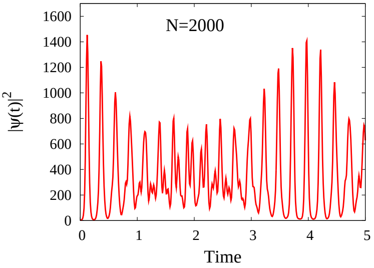
How good is semiclassics?

From eigenstates $H|\Psi_q\rangle = E_q|\Psi_q\rangle$:



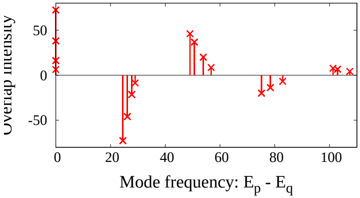
How good is semiclassics?

From eigenstates $H|\Psi_q\rangle = E_q|\Psi_q\rangle$:



If periodic,

$$\Delta E_q = q\Omega, \quad \Omega = \pi g \frac{\sqrt{N}}{\ln(\sqrt{N})}$$

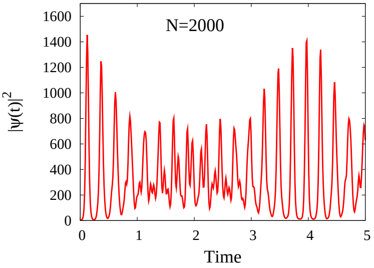


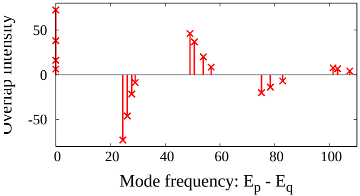
How good is semiclassics?

If periodic,

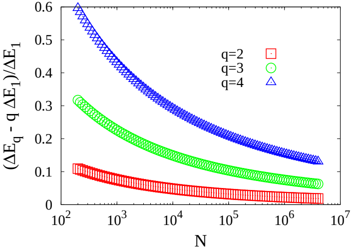
$$\Delta E_q = q\Omega, \quad \Omega = \pi g \frac{\sqrt{N}}{\ln(\sqrt{N})}$$

From eigenstates $H|\Psi_q\rangle = E_q|\Psi_q\rangle$:





Anharmonicity: $\Delta E_q - q\Delta E_1$



Semiclassical approximation: WKB quantisation

Problem is **one dimensional**; $n_{phot} + S_z \equiv N/2$, find $\Psi(n_{phot})$:

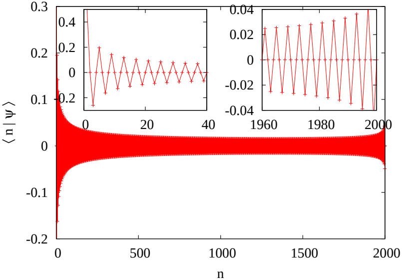
$$(E + \Delta n)\Psi_n = gn\sqrt{N+1-n}\Psi_{n-1} + g(n+1)\sqrt{N-n}\Psi_{n+1}$$

[Keeling PRA **79** 053825; see also Babelon et al. J. Stat. Mech p.07011]

Semiclassical approximation: WKB quantisation

Problem is **one dimensional**; $n_{phot} + S_z \equiv N/2$, find $\Psi(n_{phot})$:

$$(E + \Delta n)\Psi_n = gn\sqrt{N+1-n}\,\Psi_{n-1} + g(n+1)\sqrt{N-n}\,\Psi_{n+1}$$



WKB wavefunction:

$$\Psi_n \simeq \frac{\cos(E\Phi_n + \phi + n\pi/2)}{\sqrt{(n+1/2)\sqrt{N-n+1/2}}}$$

$$\Phi_n \simeq \frac{1}{g\sqrt{N+1}}\operatorname{arcosh}\left[\sqrt{\frac{N+1}{n+1/2}}\right]$$

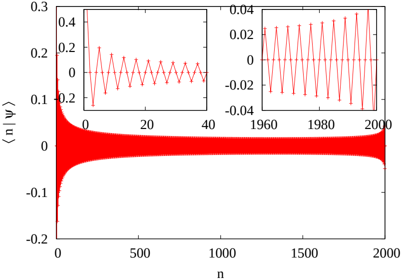
Find E, ϕ by matching assymptotics at $n \simeq 0, n \simeq N$.

[Keeling PRA **79** 053825; see also Babelon et al. J. Stat. Mech p.07011]

Semiclassical approximation: WKB quantisation

Problem is **one dimensional**; $n_{phot} + S_z \equiv N/2$, find $\Psi(n_{phot})$:

$$(E + \Delta n)\Psi_n = gn\sqrt{N+1-n}\Psi_{n-1} + g(n+1)\sqrt{N-n}\Psi_{n+1}$$



WKB wavefunction:

$$\Psi_n \simeq \frac{\cos(E\Phi_n + \phi + n\pi/2)}{\sqrt{(n+1/2)\sqrt{N-n+1/2}}}$$

$$\Phi_n \simeq \frac{1}{g\sqrt{N+1}}\text{arcosh}\left[\sqrt{\frac{N+1}{n+1/2}}\right]$$

Find E, ϕ by matching assymptotics at $n \simeq 0, n \simeq N$.

Hard boundary at $n = 0$: breakdown of Bohr-Sommerfeld quantisation.

[Keeling PRA **79** 053825; see also Babelon et al. J. Stat. Mech p.07011]

Scaling with system size

Simple approximation: match WKB soln to eqn for Ψ_0, Ψ_1:

$$1 = -\frac{2}{\pi}\sqrt{N}\frac{E}{g}\tan\left[\frac{E}{g}\frac{\ln(\sqrt{N})}{\sqrt{N}}\right],$$

Scaling with system size

Simple approximation: match WKB soln to eqn for Ψ_0, Ψ_1:

$$1 = -\frac{2}{\pi}\sqrt{N}\frac{E}{g}\tan\left[\frac{E}{g}\frac{\ln(\sqrt{N})}{\sqrt{N}}\right],$$

$$E_q = q\frac{\pi g \sqrt{N}}{\ln(\sqrt{N})} + q^3\frac{C g\sqrt{N}}{[\ln(\sqrt{N})]^4}$$

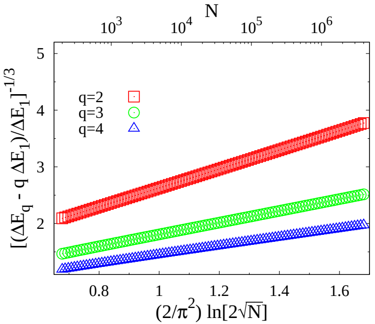
Scaling with system size

Simple approximation: match WKB soln to eqn for Ψ_0, Ψ_1:

$$1 = -\frac{2}{\pi}\sqrt{N}\frac{E}{g}\tan\left[\frac{E}{g}\frac{\ln(\sqrt{N})}{\sqrt{N}}\right],$$

$$E_q = q\frac{\pi g\sqrt{N}}{\ln(\sqrt{N})} + q^3\frac{Cg\sqrt{N}}{[\ln(\sqrt{N})]^4}$$

Semiclassics controlled by $1/\ln(N)$.



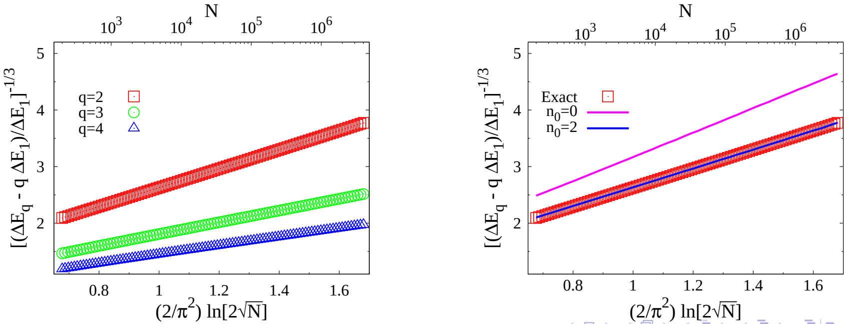
Scaling with system size

Simple approximation: match WKB soln to eqn for Ψ_0, Ψ_1:

$$1 = -\frac{2}{\pi}\sqrt{N}\frac{E}{g}\tan\left[\frac{E}{g}\frac{\ln(\sqrt{N})}{\sqrt{N}}\right],$$

$$E_q = q\frac{\pi g \sqrt{N}}{\ln(\sqrt{N})} + q^3\frac{C g\sqrt{N}}{[\ln(\sqrt{N})]^4}$$

Semiclassics controlled by $1/\ln(N)$.



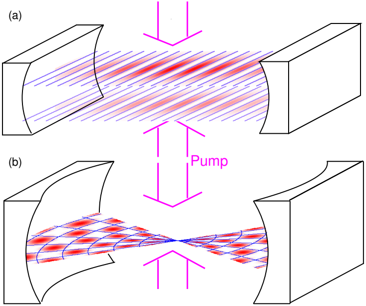
Overview

Many photon modes

nature physics
ARTICLES
PUBLISHED ONLINE: 4 OCTOBER 2009 | DOI: 10.1038/NPHYS1403

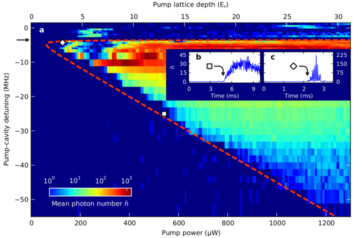
Emergent crystallinity and frustration with Bose–Einstein condensates in multimode cavities

Sarang Gopalakrishnan[1,2]*, Benjamin L. Lev[1] and Paul M. Goldbart[1,2,3]

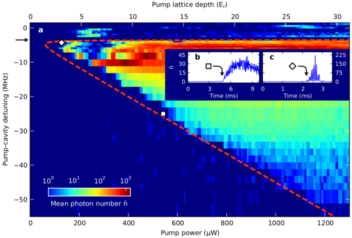


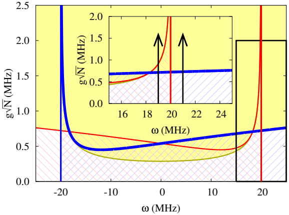
- Transition breaks $Z_2 \otimes Z_n$ — crystallisation
- No cubic mode-mode coupling — Brazovskii transition
- "Supersmectic" phase

Dynamics during/following sweep



Dynamics during/following sweep





Dynamics during/following sweep