Strong-coupling lasing and condensation

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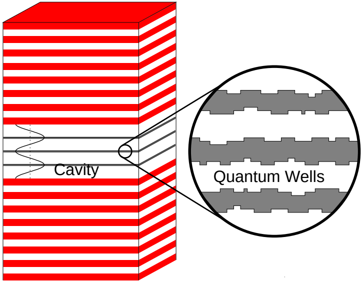
P. B. Littlewood, M. H. Szymanska.

The Burn, May 2010

Overview

1. Microcavity polariton condensation
   - Introduction to microcavity polaritons
   - A model of non-equilibrium polariton condensation
2. Strong-coupling lasing and condensation
   - Polariton condensation vs lasing
   - Why is it surprising
3. Model ingredients for strong-coupling lasing

Microcavity Polaritons



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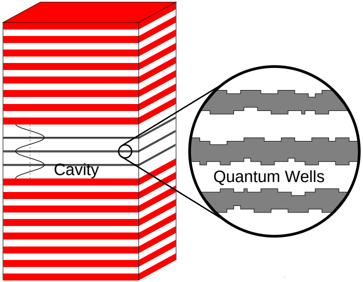
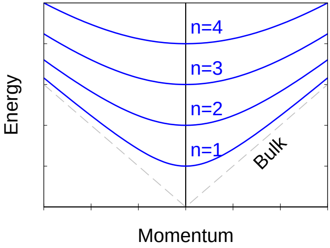


Cavity photons:

$$\omega_k = \sqrt{\omega_0^2 + c^2 k^2}$$
$$\simeq \omega_0 + k^2/2m^*$$
$$m^* \sim 10^{-4} m_e$$



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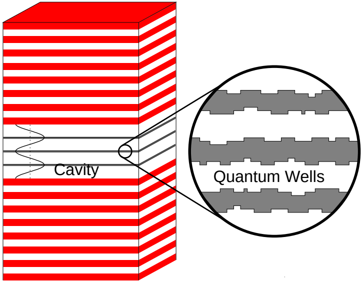

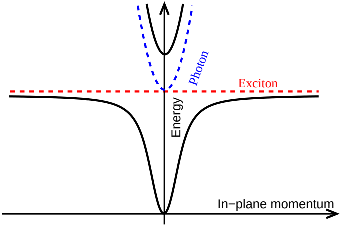

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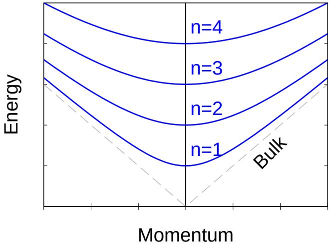
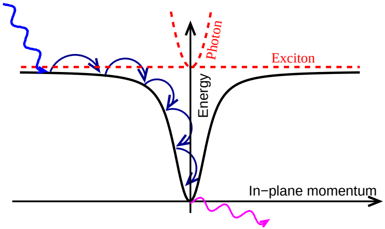


Non-equilibrium system



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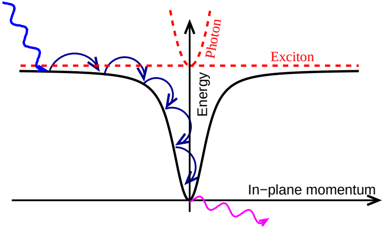

	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

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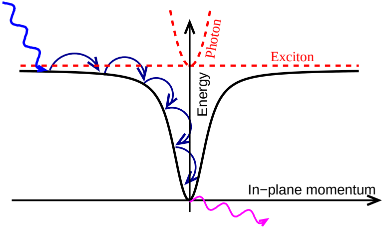

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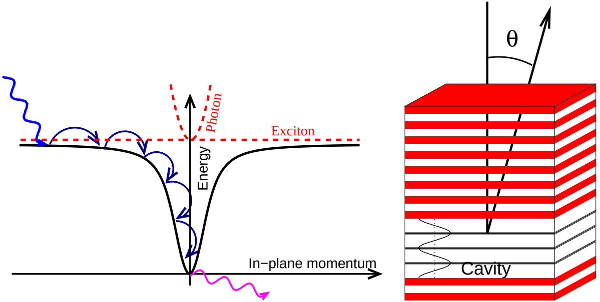

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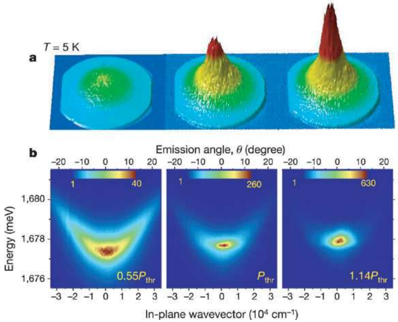
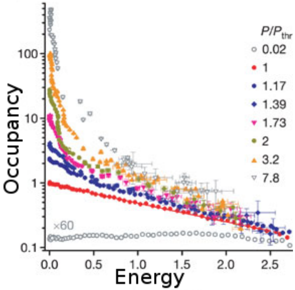
Polariton experiments: Momentum/Energy distribution





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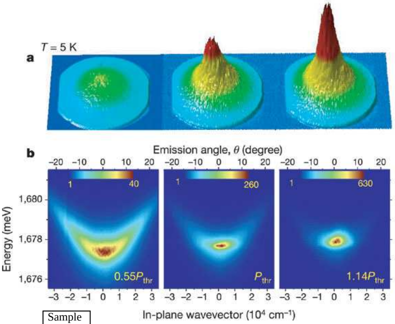
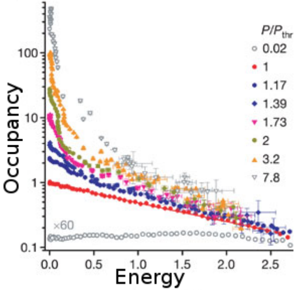
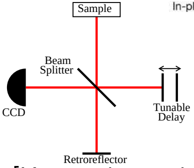
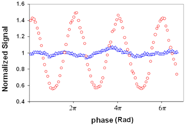
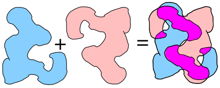


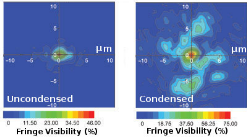






Coherence map:





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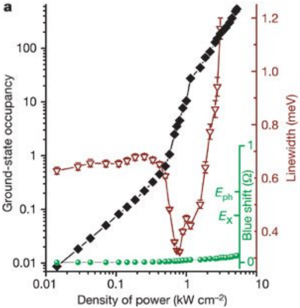
Polariton experiments: Strong coupling



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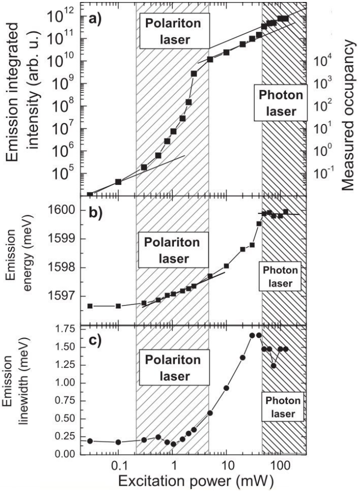

[Bajoni *et al* PRL 2008]

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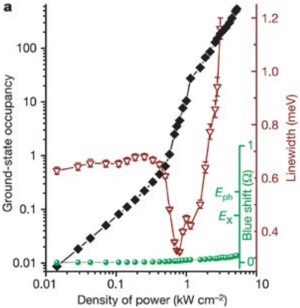

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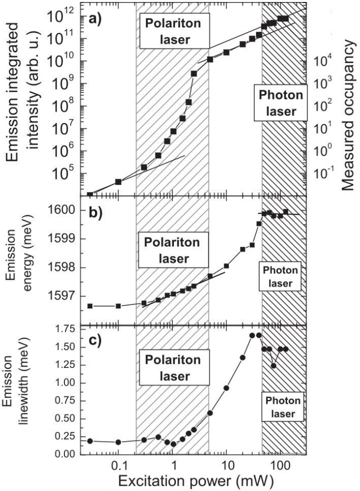
Strong coupling via:

- Small blueshift compared to Ω_R
- Polaritonic dispersion, $m > m_{phot}$
- Separate photon threshold



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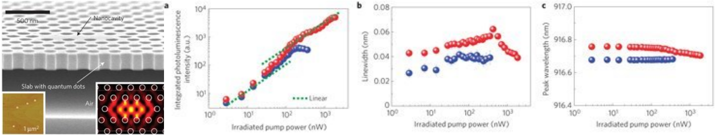
Other systems

- Quantum dot/photonic lattice (single "atom")



[Nomura *et al* Nature 2010]

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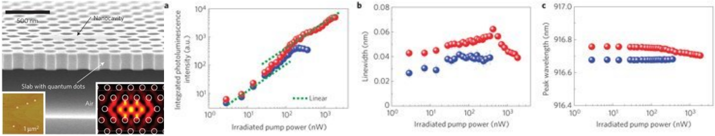

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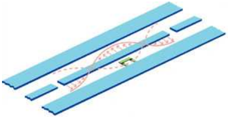

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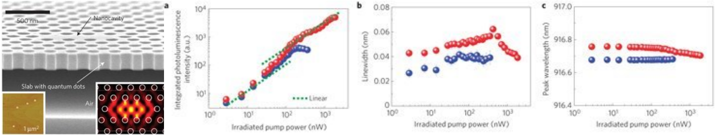

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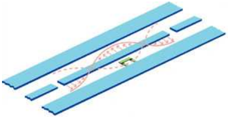

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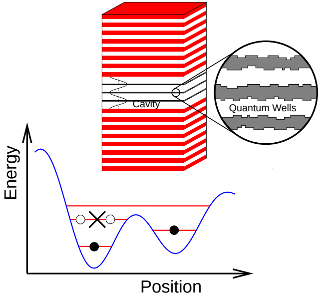
Overview

Polariton system model

Polariton model

- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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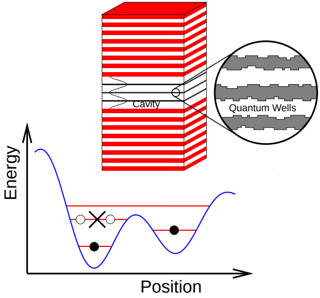
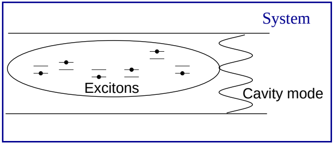


$$H_{\rm sys} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_\alpha \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{1}{\sqrt{A}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \text{H.c.} \right]$$



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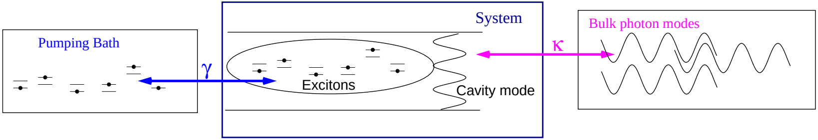

$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

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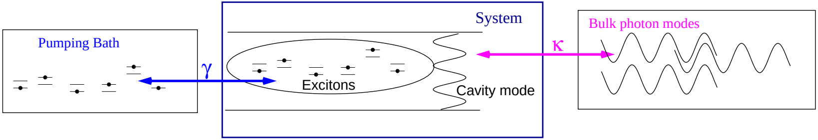

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Schematically: pump γ, decay κ

$$H_{\text{sys,bath}} \simeq \sum_{\mathbf{p},\vec{k}} \sqrt{\kappa}\psi_{\mathbf{k}}\Psi^{\dagger}_{\mathbf{p}} + \sum_{\alpha,\beta} \sqrt{\gamma}\left(a^{\dagger}_{\alpha}A_{\beta} + b^{\dagger}_{\alpha}B_{\beta}\right) + \text{H.c.}$$

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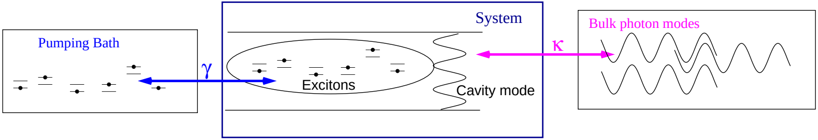

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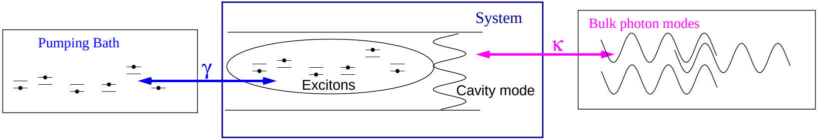

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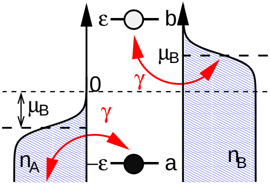
Ψ bath is empty. Pumping bath thermal, μ_B, T:



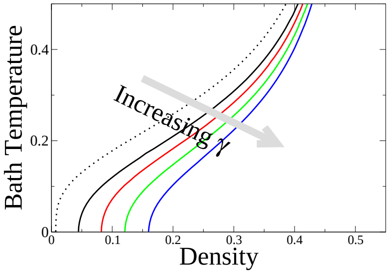
Phase boundary

1. Look for coherent solution:

$$\psi(\mathbf{r}, t) = \psi_0 e^{-i\mu_S t}.$$

Gap equation:

$$\begin{aligned}(\mu_s - \omega_0 + i\kappa)\,\psi_0 &= \frac{1}{\sqrt{A}} \sum_\alpha g_\alpha \langle P_\alpha \rangle \\ &= \chi(\psi_0, \mu_s)\psi_0\end{aligned}$$



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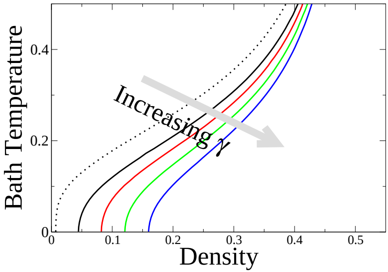

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$$\psi(t) \rightarrow \psi_0 + \delta\psi(t)$$

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grow or decay?

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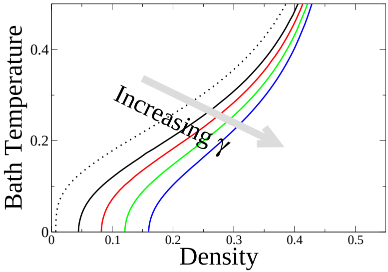

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Since non-equilibrium: Need

- Spectrum
- Occupation

Fluctuations → Stability, Luminescence, Absorption

Green's functions:

$$D^R = i\theta[t - t'] \left\langle \left[\psi, \psi^\dagger\right]_- \right\rangle$$

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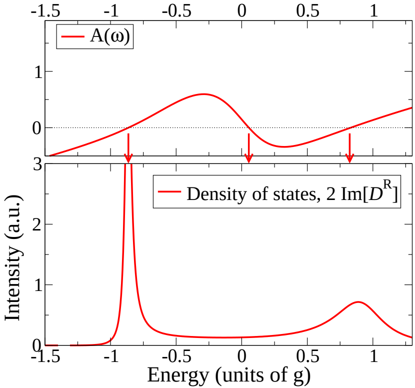

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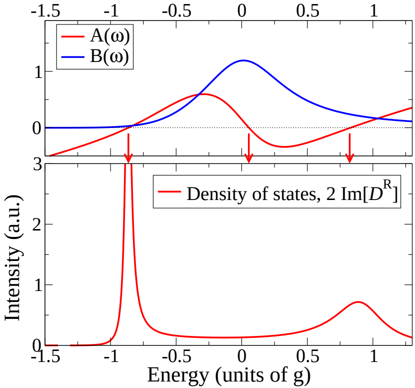

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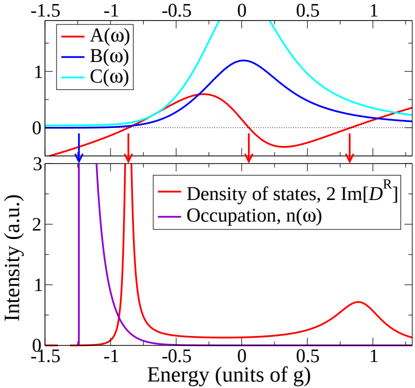

Fluctuations → Stability, Luminescence, Absorption

Green's functions:
$$D^R - D^A = -i\left\langle \left[\psi, \psi^\dagger\right]_- \right\rangle$$
$$D^K = -i\left\langle \left[\psi, \psi^\dagger\right]_+ \right\rangle = (2n(\omega) + 1)(D^R - D^A)$$

$$\left[D^R(\omega)\right]^{-1} = A(\omega) + iB(\omega),$$
$$\left[D^{-1}(\omega)\right]^K = iC(\omega),$$

$$D^K = \frac{-iC(\omega)}{B(\omega)^2 + A(\omega)^2}$$
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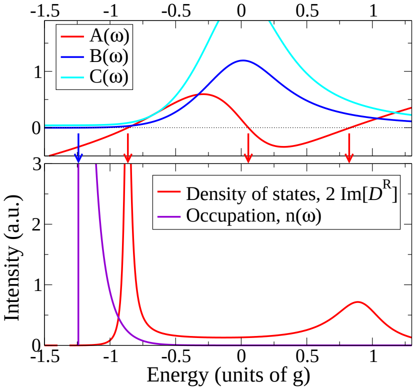
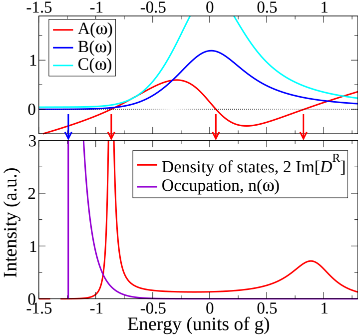
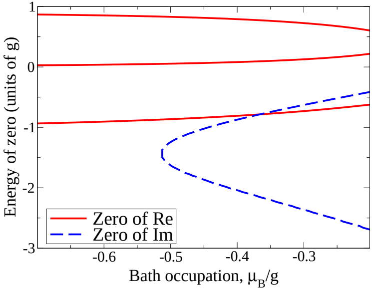
$$\left[D^R(\omega)\right]^{-1} = (\omega - \omega_k^*) + i\alpha(\omega - \mu_{\text{eff}})$$

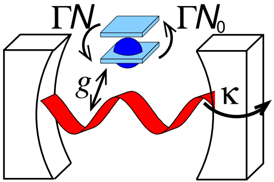


Linewidth, inverse Green's function and gap equation





$[\mathcal{D}^R]^{-1}$ for a laser



Maxwell-Bloch equations:

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
$$\partial_t P = -2i\epsilon P - \Gamma P + g\psi N$$
$$\partial_t N = \Gamma(N_0 - N) - 2g(\psi^* P + P^* \psi)$$

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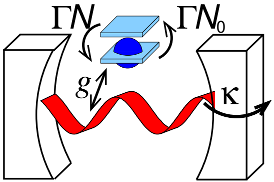

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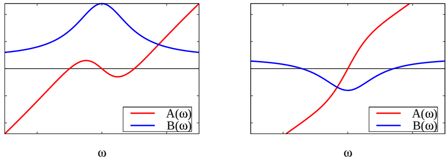

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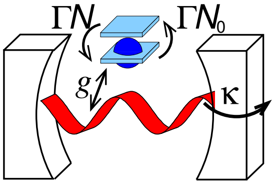

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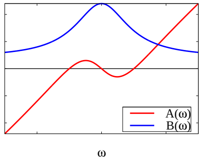
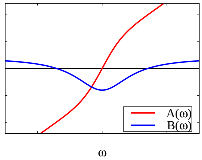
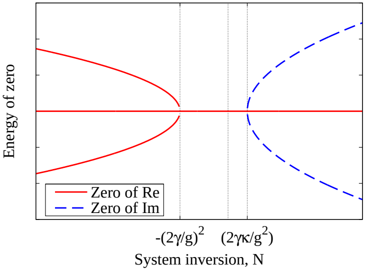






Overview

TLS Coupled to bath

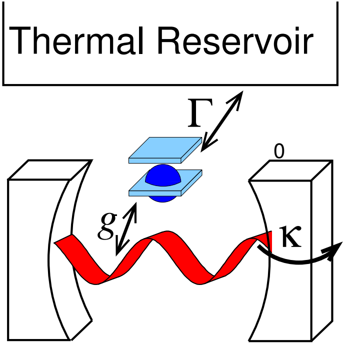
$$H = \omega_0 \psi^\dagger \psi + \sum_i \left(\epsilon \sigma_i^z + g \psi \sigma_i^+ + \text{H.c.} \right) + \sum_{n,i} \zeta_n d_n \sigma_i^+ + \text{H.c.} + \omega_n^\zeta d_n^\dagger d_n$$

Want to calculate:

$$\left[D^R(\omega) \right]^{-1} = \omega - \omega_k + i\kappa - Ng^2 \chi(\omega)$$

Bath modifies susceptibility:

$$\chi(\tau) = i\theta(\tau) \langle [\sigma^-(\tau), \sigma^+(0)] \rangle$$



Requirements for strong-coupling lasing

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Bath properties

$$d^R(\tau) = \sum_n \zeta_n^2 e^{i\omega_n^\zeta \tau} \theta(\tau)$$

$$d^K(\tau) = \sum_n \zeta_n^2 e^{i\omega_n^\zeta \tau} (2 n_B(\omega_n^\zeta) + 1)$$

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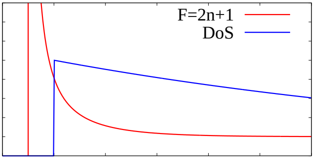

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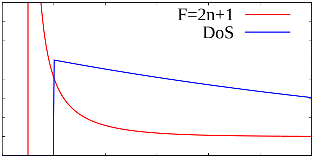

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Acknowledgements

People:

Funding:

EPSRC
Engineering and Physical Sciences Research Council

Pembroke College

Extra slides

4 Equilibrium results

5 Mean-field Keldysh theory

6 Condensed spectrum

Modelling non-equilibrium two-level system

- If $\sigma^z = \frac{1}{2}\left(b^\dagger b - a^\dagger a\right), \quad \sigma^+ = b^\dagger a$:
  Too many states $|00\rangle, |10\rangle \equiv\uparrow, |01\rangle \equiv\downarrow, |11\rangle$.
- In equilibrium, $\mathcal{Z} = \mathrm{Tr}(\rho)$
  Remove unphysical states by $\rho \to \rho \exp\left(i\frac{\pi}{2}\left[a^\dagger a + b^\dagger b\right]\right)$
- Non-equilibrium approach based on

$$\left\langle T_c\left(\psi(t,b)\psi^\dagger(0,f)\right)\right\rangle = \mathrm{Tr}\left(T_c\left[\tilde{\psi}(t,b)\tilde{\psi}^\dagger(0,f)U(-\infty,-\infty)\rho\right]\right)$$

  with

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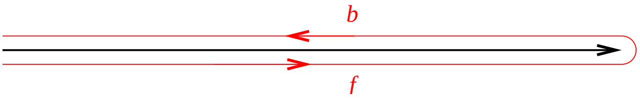
with 

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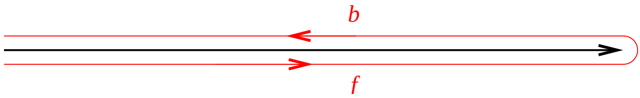
with 

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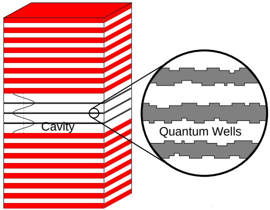
Excitons in a disorderd Quantum well



Exciton states in disorder:

$$\left[-\frac{\nabla^2_{\mathbf{R}}}{2m_X} + V(\mathbf{R})\right]\Phi_\alpha(\mathbf{R}) = \varepsilon_\alpha \Phi_\alpha(\mathbf{R})$$

$V(\vec{R})$ smoothed by exciton Bohr radius

[PRL 96 066405 (2006); PRB 76 115326 (2007)]

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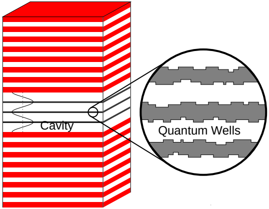

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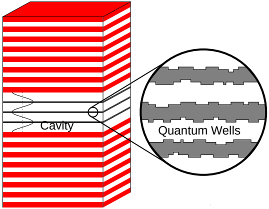

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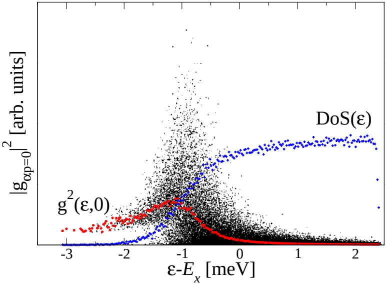

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Fluctuation corrections to phase boundary

Fluctuation corrections to density

$$\rho \rightarrow \rho_0 + \sum_{q} \langle \delta\psi^\dagger \delta\psi \rangle$$

In 2D system: modified critical condition:

$$\rho_s = \rho - \rho_{\text{normal}} = \# \frac{2m^2}{\hbar} k_B T$$

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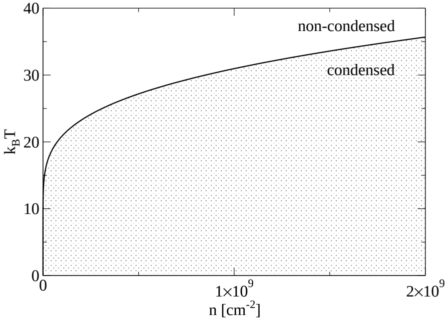

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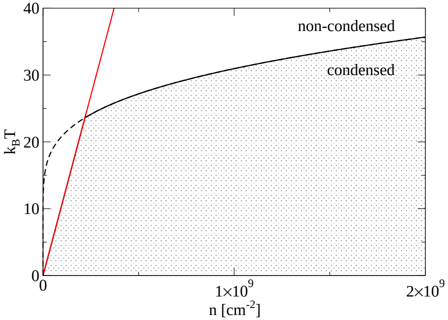

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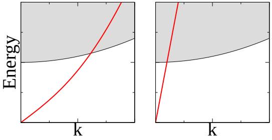

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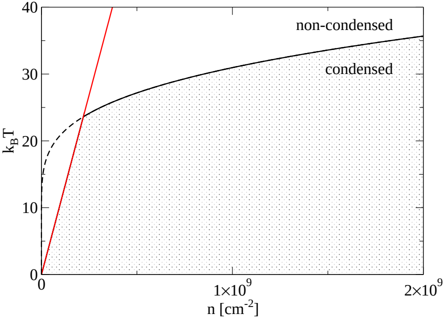

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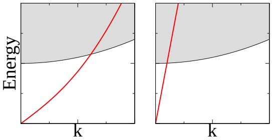

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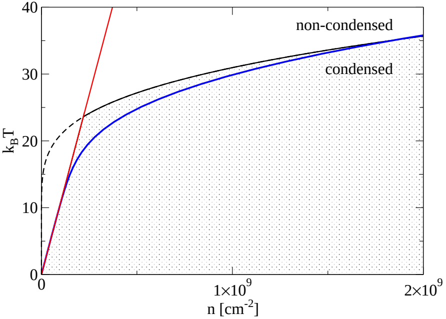

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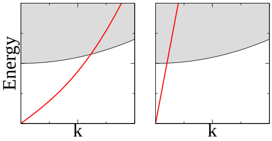

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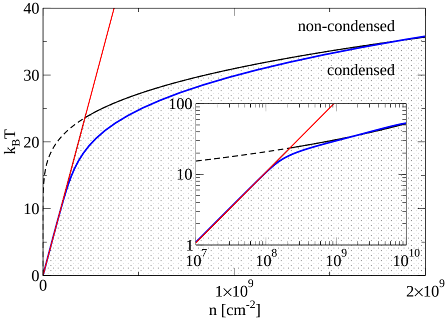

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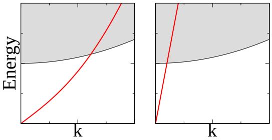

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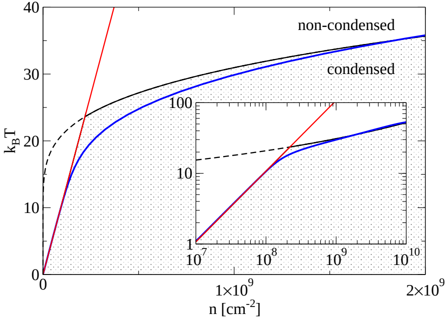

Second BCS crossover at $na_B^2 \simeq 1$

Blueshfit and experimental phase boundary

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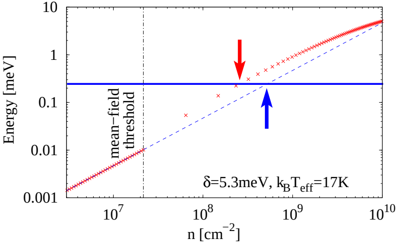

Clean limit:

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Here: $\Omega_{\mathrm{R}} a_{\mathrm{X}}^2 \to \Omega_{\mathrm{R}} \xi^2$
[PRB 77 235313]

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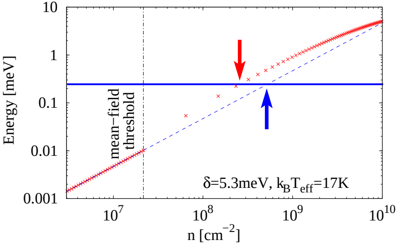

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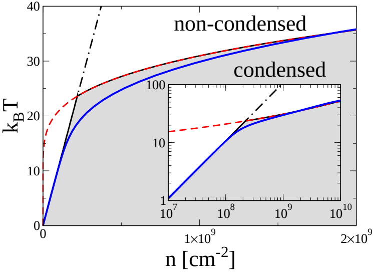
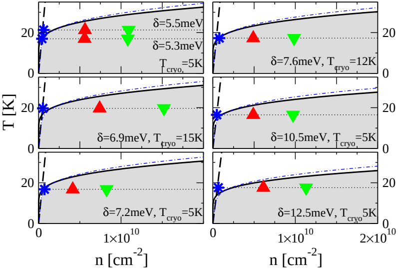
Phase diagram:

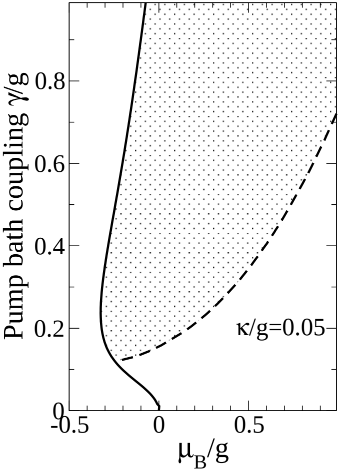


CdTe:



Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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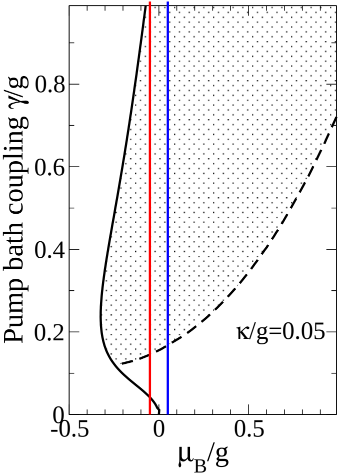

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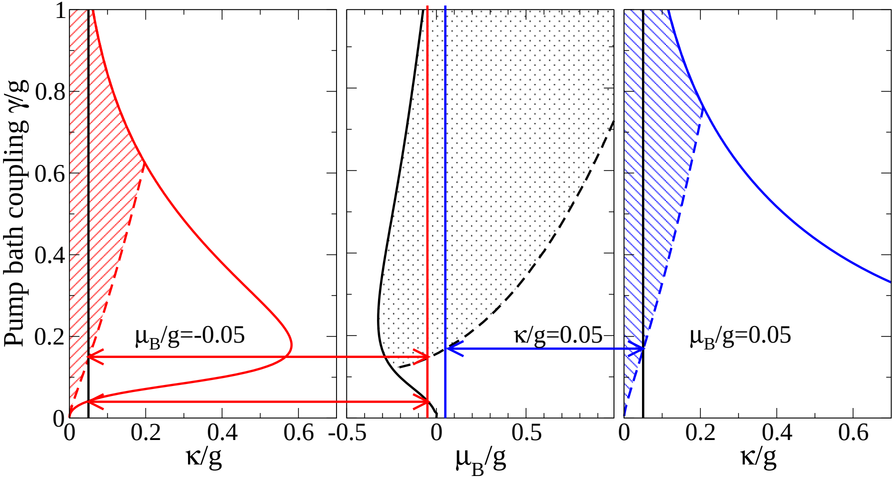


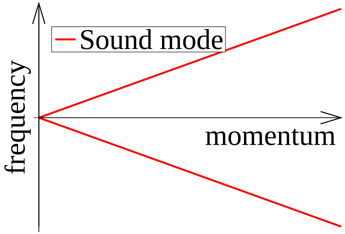
Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega, \mathbf{k})\right]^{-1} = \omega^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

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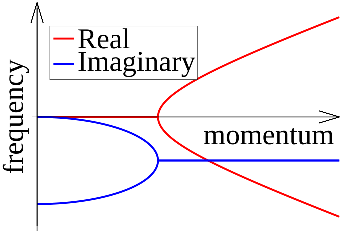

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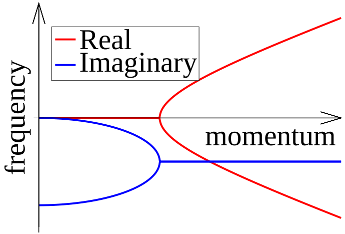

Correlations (in 2D): $\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$

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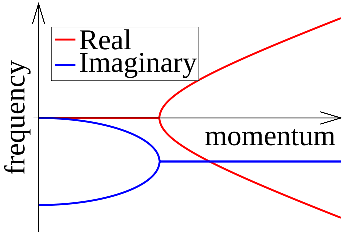

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$$\left\langle\psi^\dagger(\mathbf{r},t)\psi(0,0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta\begin{cases}\ln(r/\xi) & r\to\infty, t\simeq 0\\ \frac{1}{2}\ln(c^2t/x\xi^2) & r\simeq, t\to\infty\end{cases}\right]$$

[Szymańska et al., PRL '06; PRB '07]

Finite size effects: Single mode vs many mode

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$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_{n}^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{|(\omega+ix)^2 + x^2 - (n\Delta)^2|^2}$$

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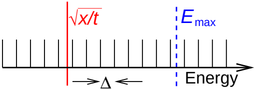
$\Delta \ll \sqrt{x/t} \ll E_{\max}$



$\mathcal{D}_{\phi\phi} \sim 1 + \ln\left(E_{\max}\sqrt{t/x}\right)$

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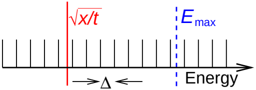

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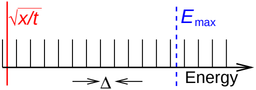

$\mathcal{D}_{\phi\phi} \sim \left(\frac{\pi C}{2x}\right)\left(\frac{t}{2x}\right)$

Relating finite-size spectrum to self phase modulation

Single mode spectrum:

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Connection to Kubo Formula [Eastham and Whittaker, arXiv:0811.4333]

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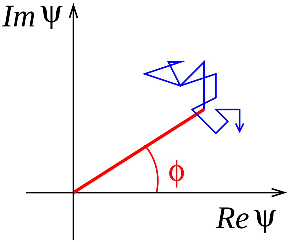

$$\partial_t \phi = U \delta N$$

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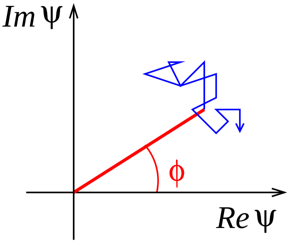

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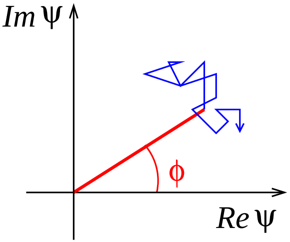

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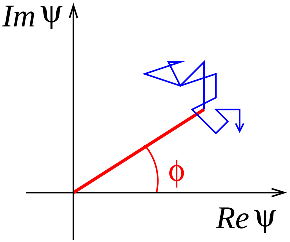

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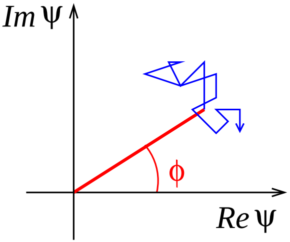

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