Quantum corrections to semiclassical dynamics in few-mode systems

Jonathan Keeling, Felix Nissen

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Tavis-Cummings model: semiclassics

Tavis-Cummings model Hamiltonian:

$$H = \sum_{i}^{N} \left(\Delta s_i^z + g(s_i^+ \psi + s_i^- \psi^\dagger) \right)$$

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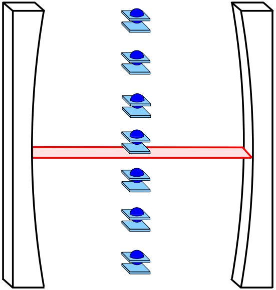

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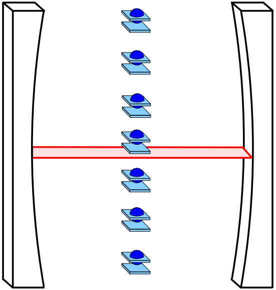

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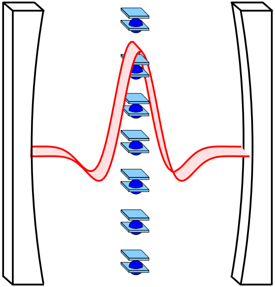

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Semiclassical equations of motion:

$$i\partial_t \psi = \sum_i^N g s^-, \quad i\partial_t s_i^- = \Delta s_i^- - 2g s_i^z \psi, \quad i\partial_t s_i^z = g(s_i^+ \psi - s_i^- \psi^*)$$

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Can be solved for s in terms of
$\lambda = N - (\Delta/2g)^2$

$$\dot{\psi}^2 = g^2 \psi^2 (\lambda - \psi^2),$$

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$$\psi(t) \simeq \sum_m \frac{\sqrt{\lambda}}{\cosh[g\sqrt{\lambda}(t - mT)]},$$

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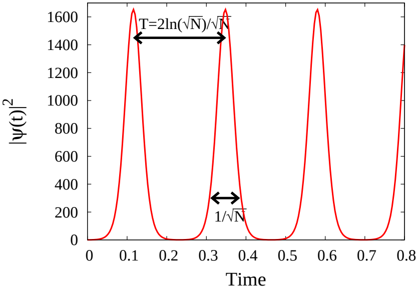

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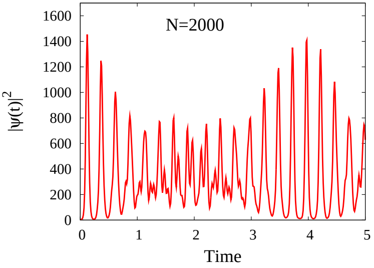

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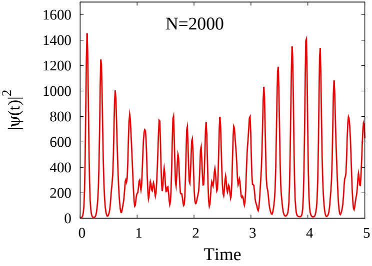
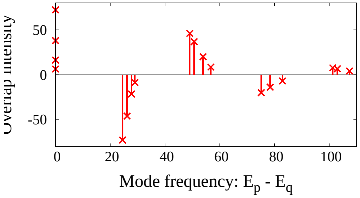


If periodic,

$$\Delta E_q = q\Omega, \quad \Omega = \pi g \frac{\sqrt{N}}{\ln(\sqrt{N})}$$



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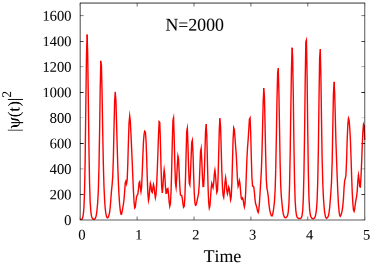
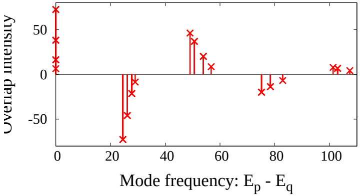
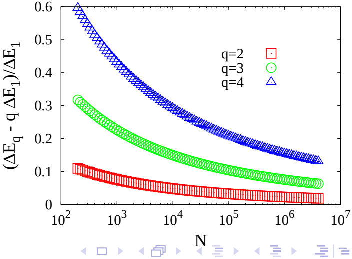




Anharmonicity: $\Delta E_q - q\Delta E_1$



Semiclassical approximation: WKB quantisation

Problem is **one dimensional**; $n_{phot} + S_z \equiv N/2$, find $\Psi(n_{phot})$:

$$(E + \Delta n)\Psi_n = gn\sqrt{N+1-n}\Psi_{n-1} + g(n+1)\sqrt{N-n}\Psi_{n+1}$$

[Keeling PRA **79** 053825; see also Babelon et al. J. Stat. Mech p.07011]

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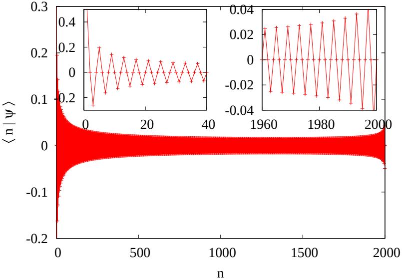

WKB wavefunction:

$$\Psi_n \simeq \frac{\cos(E\Phi_n + \phi + n\pi/2)}{\sqrt{(n+1/2)\sqrt{N-n+1/2}}}$$

$$\Phi_n \simeq \frac{1}{g\sqrt{N+1}} \text{arcosh}\left[\sqrt{\frac{N+1}{n+1/2}}\right]$$

Find E, ϕ by matching assymptotics at $n \simeq 0, n \simeq N$.

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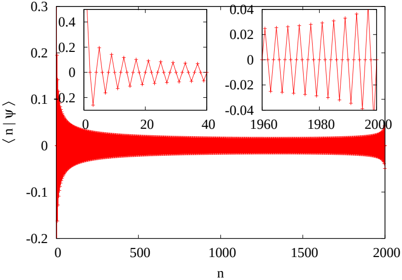

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Hard boundary at $n = 0$: breakdown of Bohr-Sommerfeld quantisation.
[Keeling PRA **79** 053825; see also Babelon et al. J. Stat. Mech p.07011]

Scaling with system size

Simple approximation: match WKB soln to eqn for Ψ_0, Ψ_1:

$$1 = -\frac{2}{\pi}\sqrt{N}\frac{E}{g}\tan\left[\frac{E}{g}\frac{\ln(\sqrt{N})}{\sqrt{N}}\right],$$

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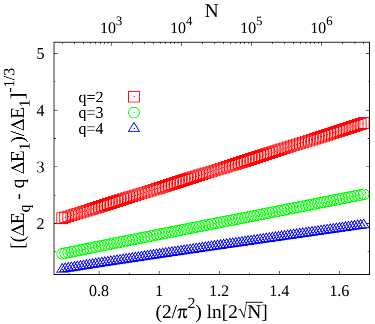

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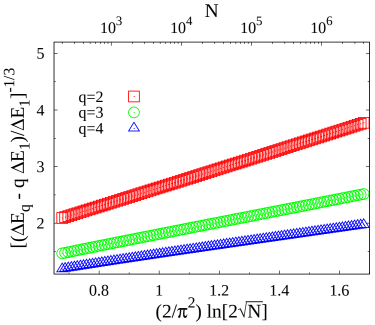
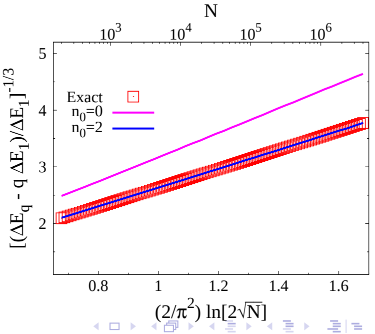
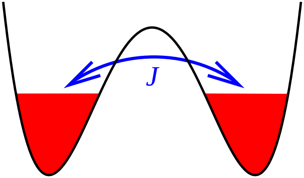




Josephson model



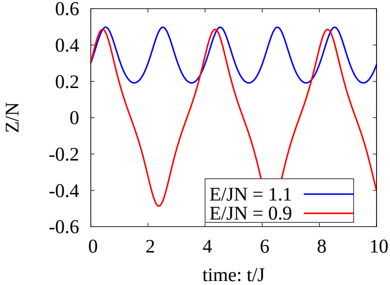
Spatially separted states:

$$H = U\left(\Psi_L^\dagger\Psi_L - \Psi_R^\dagger\Psi_R\right)^2 + J\left(\Psi_L^\dagger\Psi_R + \Psi_R^\dagger\Psi_L\right)$$

Fixed # atoms, $N = \Psi_L^\dagger\Psi_L + \Psi_R^\dagger\Psi_R$

Classical dynamics of $Z = \frac{1}{2}\left(\Psi_L^\dagger\Psi_L - \Psi_R^\dagger\Psi_R\right)$

- If $2NU > J$, then:
  - $E < NJ$ untrapped oscillations
  - $E > NJ$ self-trapped oscillations
- If $2NU < J$, untrapped oscillations



1D Quantum mechanics problem and WKB solution

$$(E - Un^2)\psi_n = \frac{J}{2}\left[\sqrt{(N+n)(N-n+2)}\psi_{n-2} + \sqrt{(N-n)(N+n+2)}\psi_{n+2}\right]$$

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For $z = n/N$, allowed/forbidden regions:

$$\psi(z) = b(z)\begin{cases} C_+ e^{iNW_0 + iW_1} + \dots \\ C_+ e^{N\Omega_0 + \Omega_1} + \dots \end{cases}$$

$$\cos(2W_0') = \frac{\epsilon - uz^2}{\sqrt{1-z^2}} = \cosh(2\Omega_0')$$

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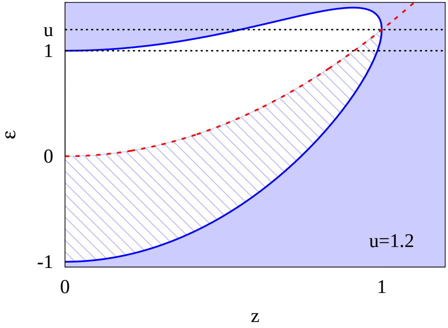
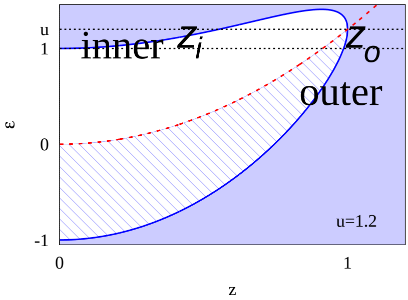


Level spacing and WKB result



- $\epsilon < 1 \qquad \int_{-z_o}^{z_o} dz[NW_0' + W_1'] = m\pi + \frac{\pi}{2}$
- $\epsilon > 1 \qquad \int_{z_i}^{z_o} dz[NW_0' + W_1'] \simeq m\pi + \frac{\pi}{2} \pm \exp\left(-\int_{-z_i}^{z_i} dz[N\Omega_0' + \Omega_1']\right)$
- $\epsilon \simeq 1 \qquad F\left[N(\epsilon - 1)\ln(N) \ldots\right] = m\pi + \frac{\pi}{2}$

[F.B.F. Nissen, JK, `arXiv:0912:0189`]

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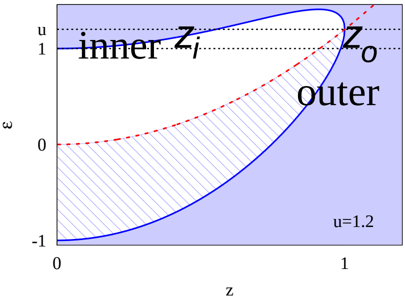

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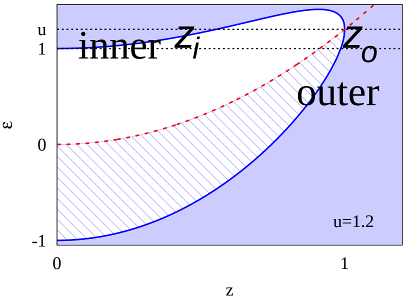

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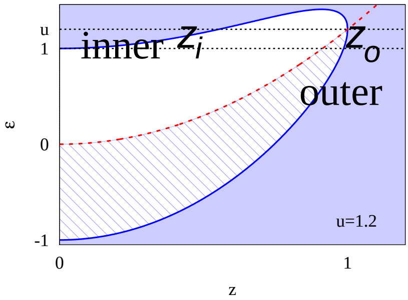

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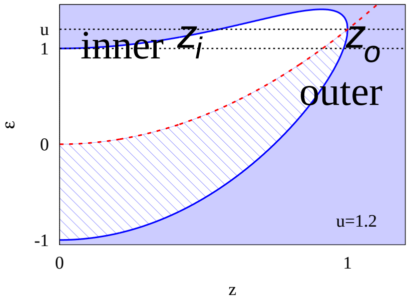

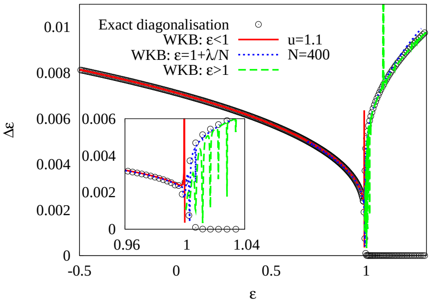

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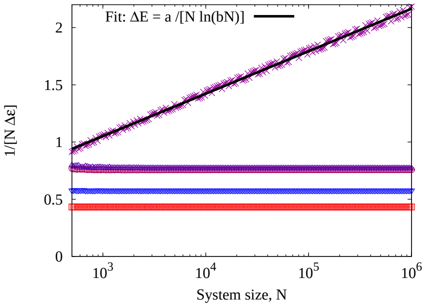
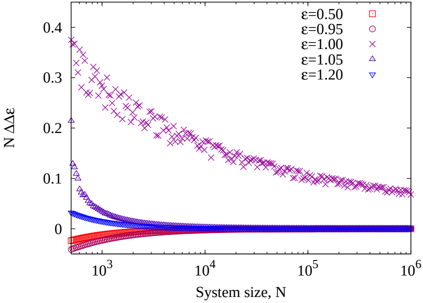
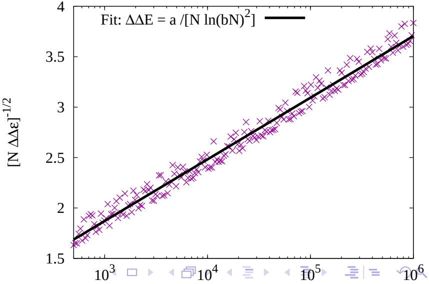
Scaling vs system size

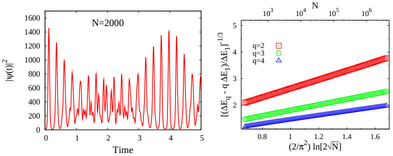






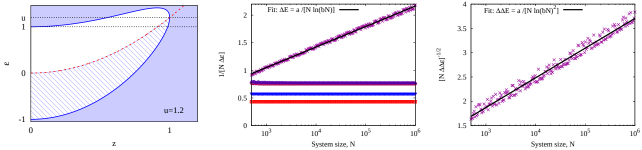
Conclusions

- Examples of logarithmic scaling in many-body QM problems
- Tavis-Cummings model — significant corrections for frequently studied initial conditions



[JK, PRA **79** 053825]

- Tavis-Cumings — Critical level due to end point
- Josephson model — Critical level due to bifurcation



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Extra slides