Nonequilibrium quantum condensates: from microscopic theory to macroscopic phenomenology

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Acknowledgements

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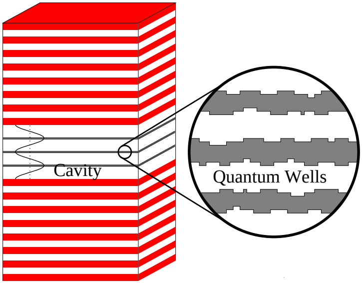
Microcavity Polaritons



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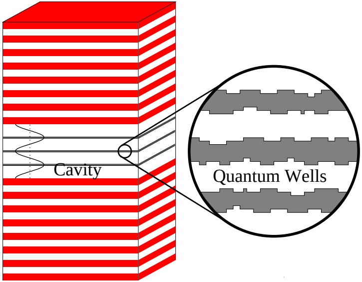

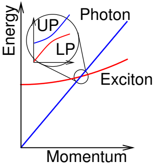

[Pekar, JETP(1958)]
[Hopfield, Phys. Rev.(1958)]

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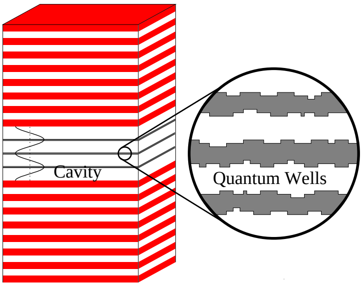

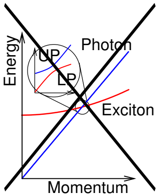

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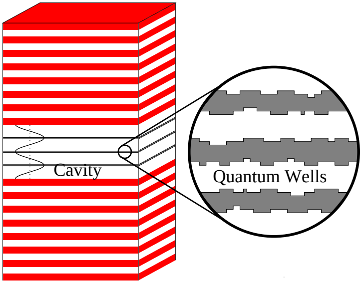

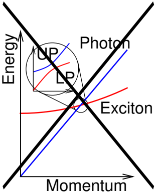

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Cavity photons:

$$\omega_k = \sqrt{\omega_0^2 + c^2 k^2}$$
$$\simeq \omega_0 + k^2/2m^*$$
$$m^* \sim 10^{-4} m_e$$

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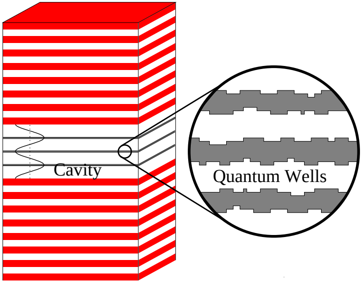

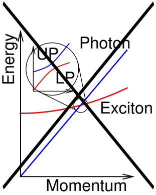

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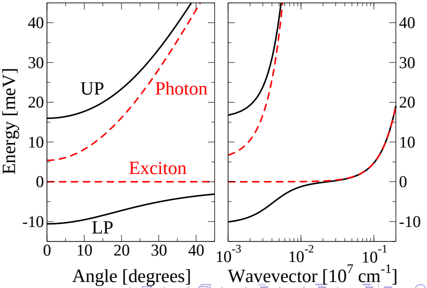
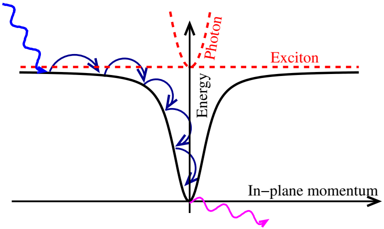
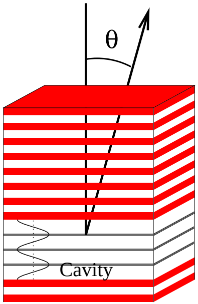
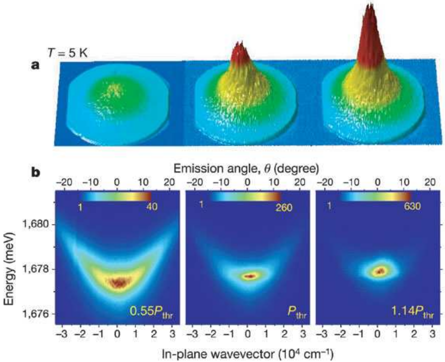


Non-equilibrium: flux and baths

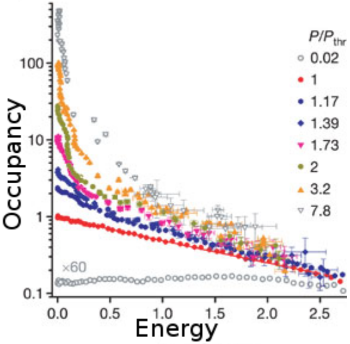




Polariton experiments: Momentum/Energy distribution

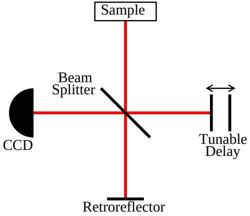


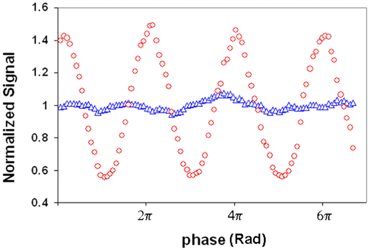
[Kasprzak, et al., Nature, 2006]



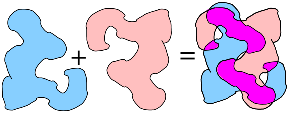
Polariton experiments: Coherence

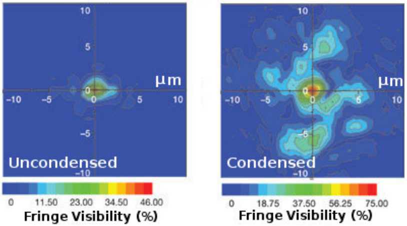
Basic idea:





Coherence map:

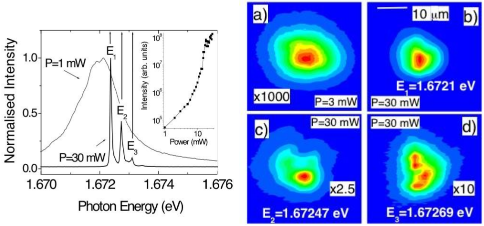


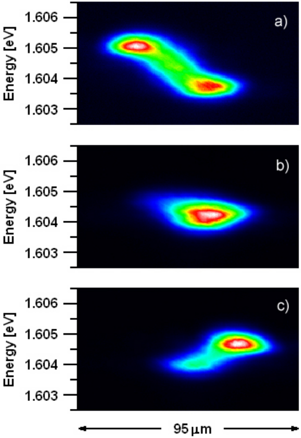


[Kasprzak, et al., Nature, 2006]

Other polariton condensation experiments

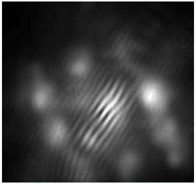
- Stress traps for polaritons
  [Balili *et al* Science 316 1007 (2007)]
- Temporal coherence and line narrowing
  [Love *et al* Phys. Rev. Lett. 101 067404 (2008)]



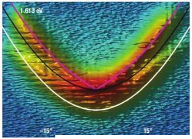


Other polariton condensation experiments

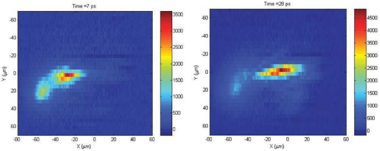
- Quantised vortices in disorder potential
  [Lagoudakis *et al* Nature Phys. 4, 706 (2008)]



- Changes to excitation spectrum
  [Utsunomiya *et al* Nature Phys. 4 700 (2008)]



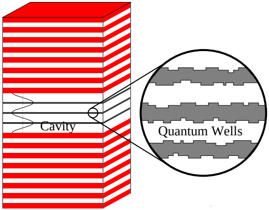
- Soliton propagation
  [Amo *et al* Nature 457 291 (2009)]
- Driven superfluidity
  [Amo *et al* Nature Phys. (2009)]



Overview

1. Introduction to microcavity polaritons
2. Model and review of equilibrium results
   - Disorder-localised exciton model
   - Equilibrium mean field theory
3. Microscopic non-equilibrium model
   - Model and mean-field theory
   - Fluctuations
     - Stability of normal state — lasing vs condensation
     - Condensed spectrum
4. Macroscopic phenomenology
   - Gross Pitaevskii equation in an harmonic trap
   - Internal Josephson effect and spatial variation
     - Spin degree of freedom
     - Spin and spatial degrees of freedom
5. Conclusions

Excitons in a disorderd Quantum well

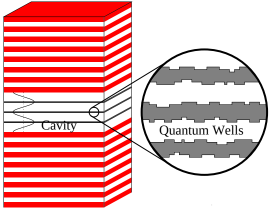


Exciton states in disorder:

$$\left[-\frac{\nabla^2_{\mathbf{R}}}{2m_{\mathrm{X}}} + V(\mathbf{R})\right]\Phi_\alpha(\mathbf{R}) = \varepsilon_\alpha \Phi_\alpha(\mathbf{R})$$

$V(\vec{R})$ smoothed by exciton Bohr radius

[PRL 96 066405 (2006); PRB 76 115326 (2007)]



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Want: Energies ε_α Oscillator strengths: $g_{\alpha,\mathbf{p}} \propto \psi_{1s}(0)\Phi_{\alpha,\mathbf{p}}$

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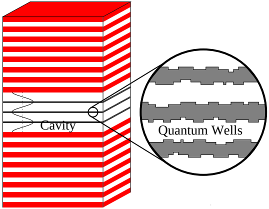

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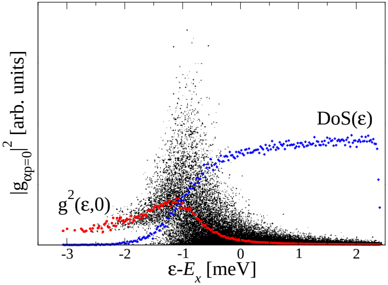

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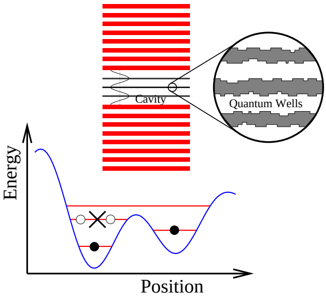
Polariton system model

Polariton model

- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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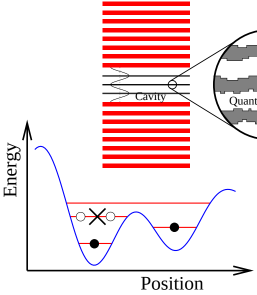
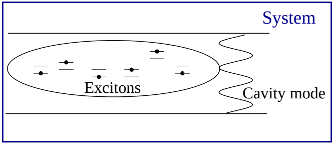
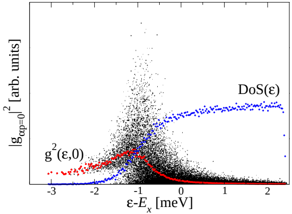


$$H_{\mathrm{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{1}{\sqrt{\mathrm{Area}}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \mathrm{H.c.} \right]$$



Equilibrium: Mean-field theory

$$H_{\text{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{g_{\alpha,\mathbf{k}}}{\sqrt{A}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \text{H.c.} \right]$$



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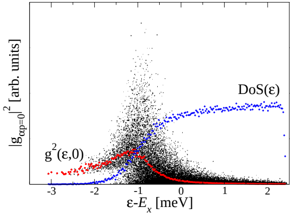
Mean-field theory:
Self-consistent polarisation and field

$$\left[i\partial_t + \omega_0 - \frac{\nabla^2}{2m} \right] \psi = \frac{1}{\sqrt{A}} \sum_\alpha g_\alpha P_\alpha$$



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Mean-field theory:

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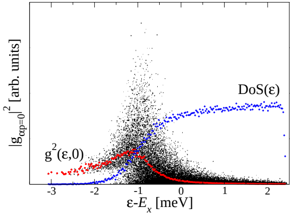
$$\left[-\mu + \omega_0 - \frac{\nabla^2}{2m} \right] \psi = \frac{1}{\sqrt{A}} \sum_\alpha g_\alpha \langle a^\dagger_\alpha b_\alpha \rangle$$



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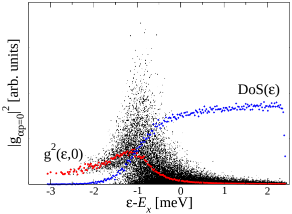
$$\left[-\mu + \omega_0 - \frac{\nabla^2}{2m} \right] \psi = \frac{1}{\sqrt{A}} \sum_\alpha g_\alpha \frac{g_\alpha \psi}{2E_\alpha} \tanh(\beta E_\alpha)$$

$$E_\alpha{}^2 = \left(\frac{\epsilon_\alpha - \mu}{2} \right)^2 + g_\alpha^2 \psi^2$$



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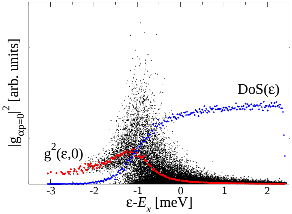

Density

$$\rho = |\psi|^2 + \frac{1}{A} \sum_{\alpha} \left[\frac{1}{2} - \frac{\epsilon_{\alpha} - \mu}{4 E_{\alpha}} \tanh(\beta E_{\alpha}) \right]$$

Equilibrium: Mean-field theory

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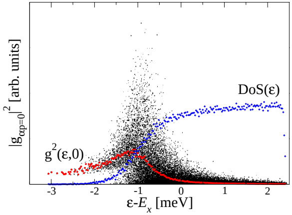

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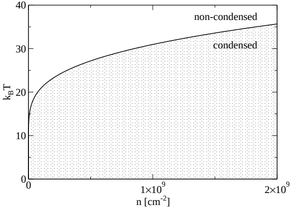
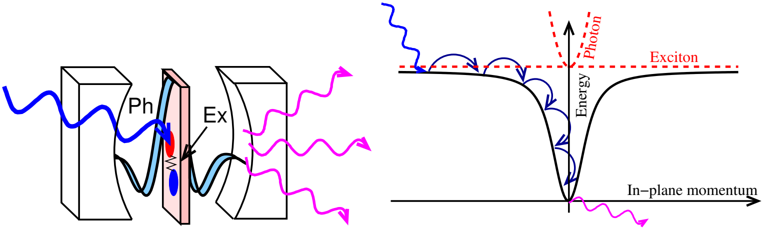


Overview

Non-equilibrium system



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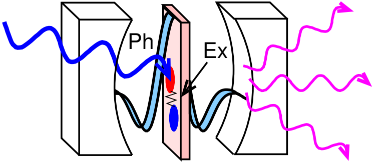
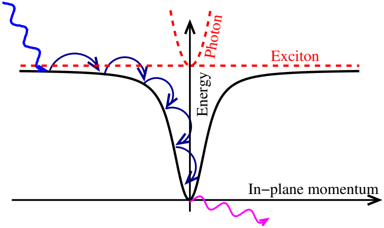
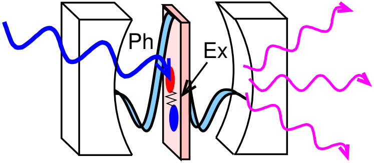


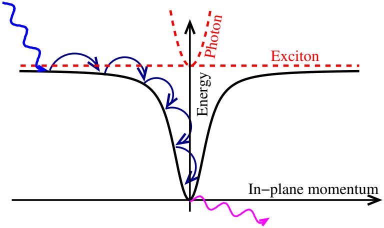


	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Non-equilibrium system

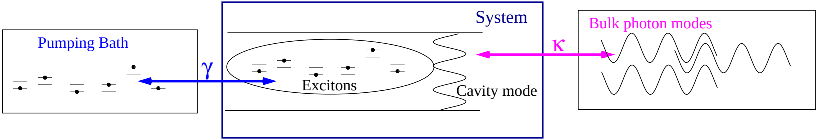




	Lifetime	Thermalisation	Linewidth	Temperature	
Atoms	10s	10ms	2.5×10^{-13}meV	10^{-8}K	10^{-9}meV
Excitons[a]	50ns	0.2ns	5×10^{-5}meV	1K	0.1meV
Polaritons	5ps	0.5ps	0.5meV	20K	2meV
Magnons[b]	1μs(??)	100ns(?)	2.5×10^{-6}meV	300K	30meV

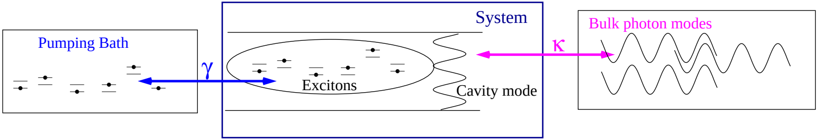
[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Non-equilibrium model: baths



$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

Non-equilibrium model: baths



$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

Schematically: pump γ, decay κ

$$H_{\text{sys,bath}} \simeq \sum_{\mathbf{p},\vec{k}} \sqrt{\kappa}\psi_{\mathbf{k}}\Psi^{\dagger}_{\mathbf{p}} + \sum_{\alpha,\beta} \sqrt{\gamma}\left(a^{\dagger}_{\alpha}A_{\beta} + b^{\dagger}_{\alpha}B_{\beta}\right) + \text{H.c.}$$

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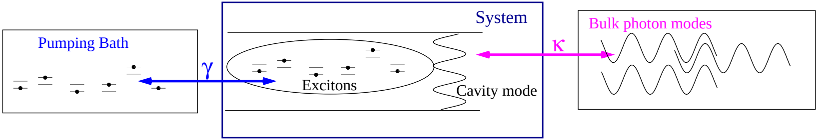

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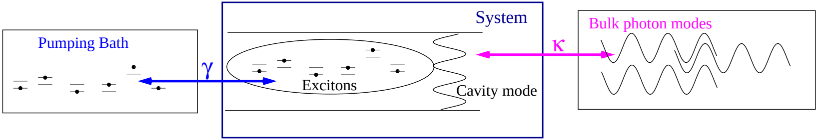
Bath correlations, $\langle \Psi^\dagger \Psi \rangle$, $\langle A^\dagger A \rangle$, $\langle B^\dagger B \rangle$ fixed:

Non-equilibrium model: baths



$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

Schematically: pump γ, decay κ

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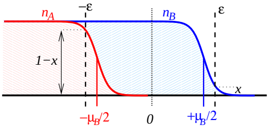
Ψ bath is empty. Pumping bath thermal, μ_B, T:



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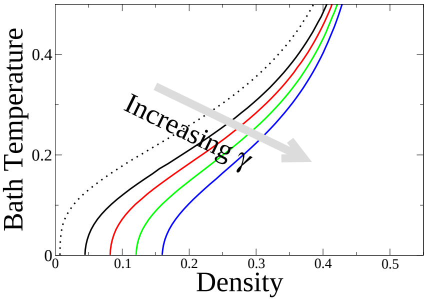

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$$\omega_0 - \mu_s = \frac{g^2}{2E} \tanh\left(\frac{\beta E}{2}\right)$$

Overview

Fluctuations → Stability, Luminescence, Absorption

Keldysh approach:

$$D^{R,A} = \mp i\theta[\pm(t-t')]\left\langle\left[\psi,\psi^\dagger\right]_-\right\rangle$$

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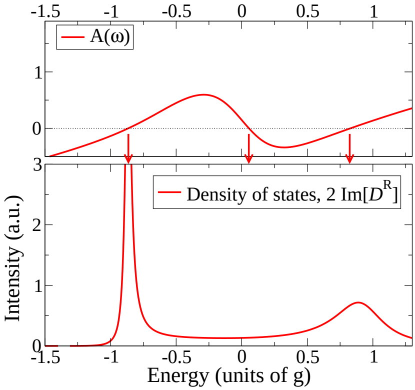

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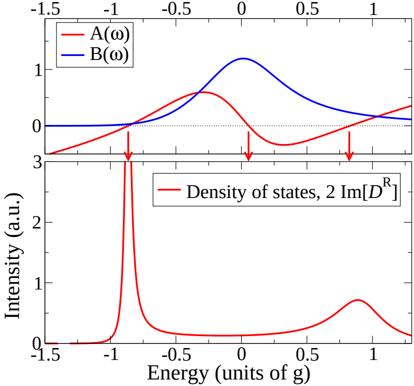

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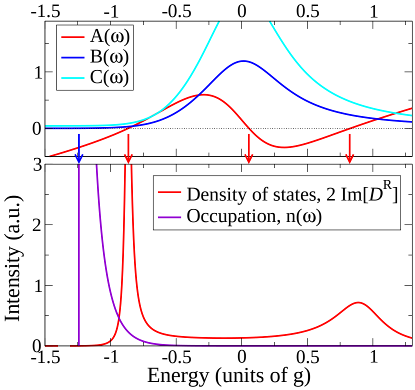

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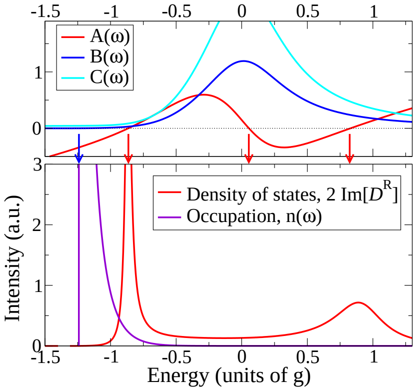
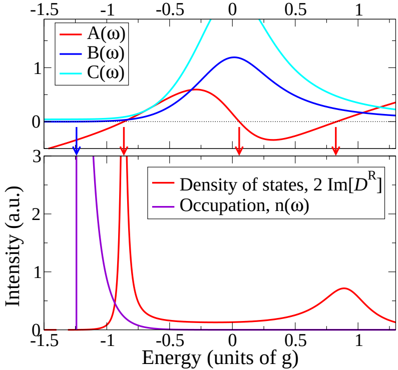


Linewidth, inverse Green's function and gap equation



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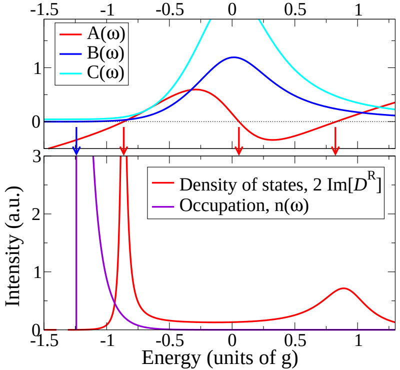
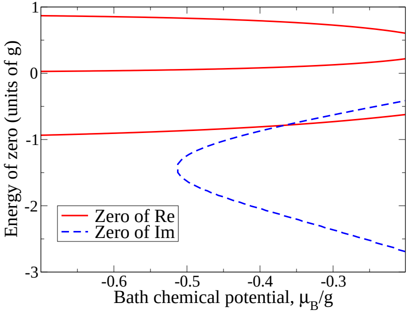


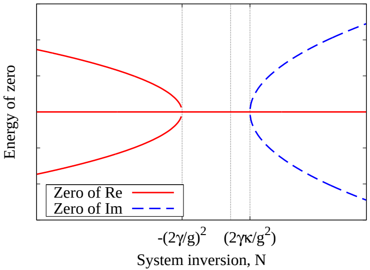


$[\mathcal{D}^R]^{-1}$ for a laser

Maxwell-Bloch equations:

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
$$\partial_t P = -2i\epsilon P - \Gamma P + g\psi N$$
$$\partial_t N = \Gamma(N_0 - N) - 2g(\psi^* P + P^* \psi)$$

$$\left[D^R(\omega)\right]^{-1} = \omega - \omega_k + i\kappa + \frac{g^2 N_0}{\omega - 2\epsilon + i\Gamma}$$



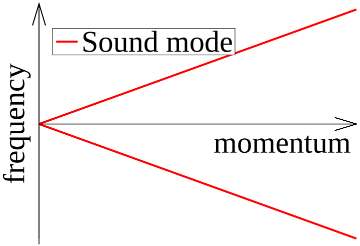
Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega, \mathbf{k})\right]^{-1} = \omega^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

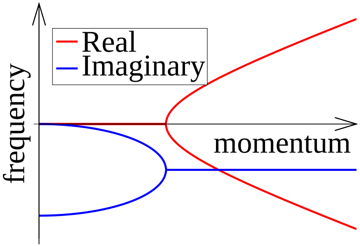
Fluctuations above transition

When condensed

$$\text{Det}\left[D^R(\omega,\mathbf{k})\right]^{-1} = (\omega+ix)^2+x^2-c^2\mathbf{k}^2$$

Poles:

$$\omega^* = -ix \pm \sqrt{c^2\mathbf{k}^2 - x^2}$$



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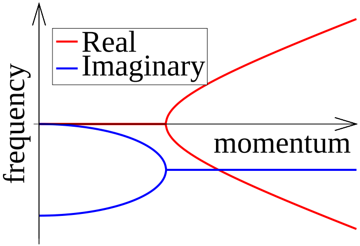

Correlations (in 2D): $\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$

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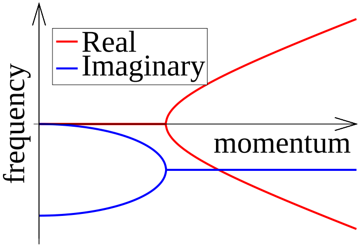

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$$\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta\begin{cases}\ln(r/\xi) & r\to\infty, t\simeq 0\\ \frac{1}{2}\ln(c^2t/x\xi^2) & r\simeq, t\to\infty\end{cases}\right]$$

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Overview

Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit:

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Nonlinear, complex susceptibility $\chi(\psi(r,t))$

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$$\left(i\hbar\partial_t + i\kappa - \left[V(r) - \frac{\hbar^2\nabla^2}{2m}\right]\right)\psi(r) = \chi(\psi(r,t))\psi(r,t)$$

Nonlinear, complex susceptibility $\chi[E(\psi(r,t))]$, $E_\alpha^2 = \epsilon_\alpha^2 + g^2|\psi_0|^2$

$$i\hbar\partial_t\psi|_{\text{nlin}} = U|\psi|^2\psi$$

$$i\hbar\partial_t\psi|_{\text{loss}} = -i\kappa\psi \qquad i\hbar\partial_t\psi|_{\text{gain}} = i\gamma_{\text{eff}}(\mu_B)\psi - i\Gamma|\psi|^2\psi$$

$$i\partial_t\psi = \left[-\frac{\hbar^2\nabla^2}{2m} + V(r) + U|\psi|^2 + i\left(\gamma_{\text{eff}}(\mu_B) - \kappa - \Gamma|\psi|^2\right)\right]\psi$$

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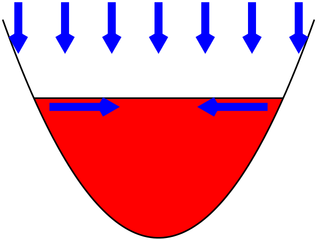

[Keeling & Berloff, PRL, '08]

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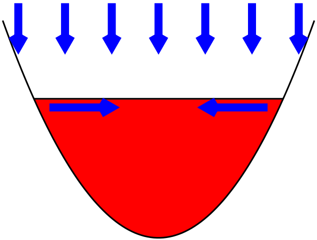
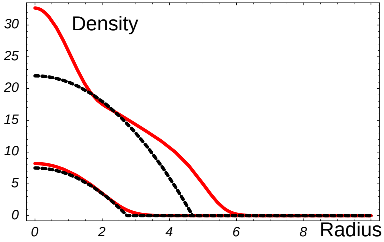
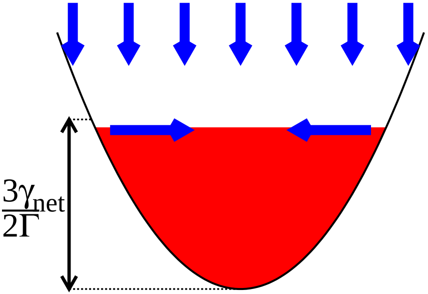




[Keeling & Berloff, PRL, '08]

Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = \frac{1}{\hbar}(\gamma_{\text{net}} - \Gamma\rho)\rho$$



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High m modes: $\delta\rho_{n,m} \simeq e^{im\theta} r^m \ldots$

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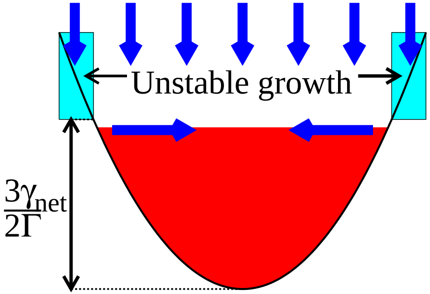

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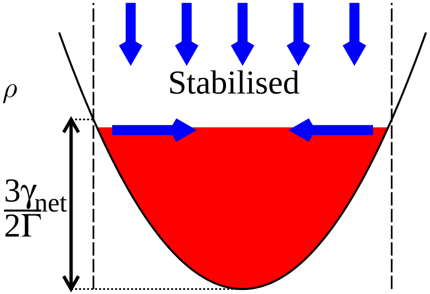

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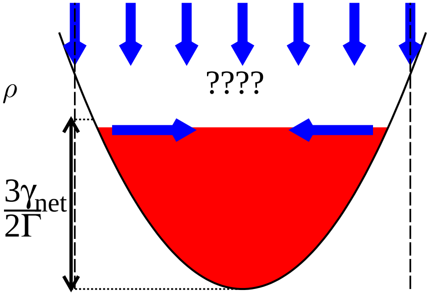
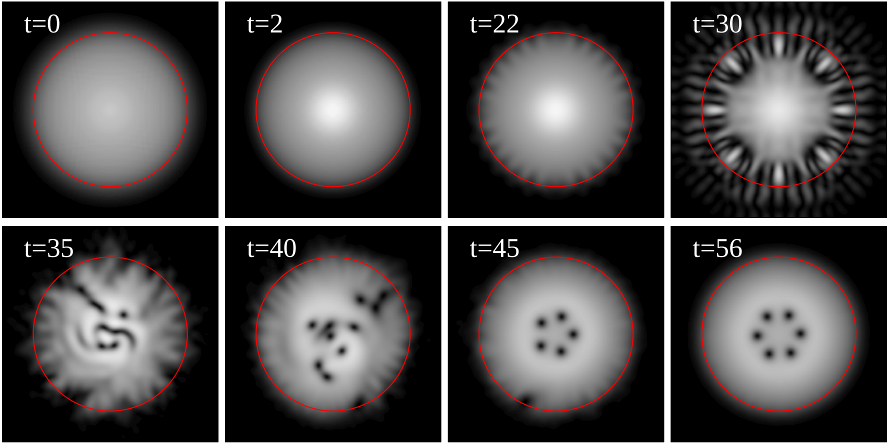
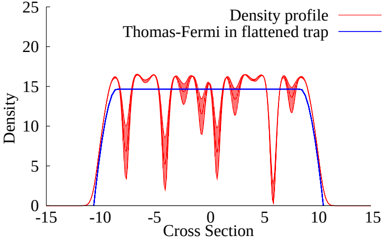


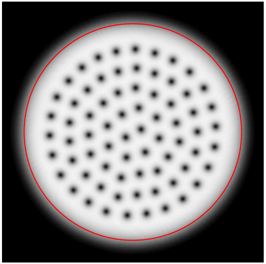
Time evolution:



[Keeling & Berloff, PRL, '08]

Why vortices





$$\nabla\cdot[\rho(\mathbf{v}-\Omega\times\mathbf{r})] = (\gamma_{\mathrm{net}}\Theta(r_0-r)-\Gamma\rho)\,\rho,$$

$$\mu = \frac{m}{2}\left|\mathbf{v}-\Omega\times\mathbf{r}\right|^2 + \frac{m}{2}r^2(\omega^2-\Omega^2) + U\rho - \frac{\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}$$

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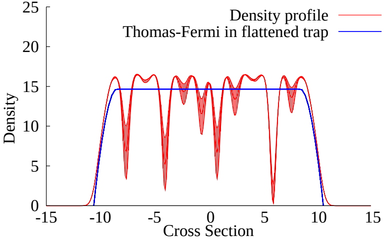

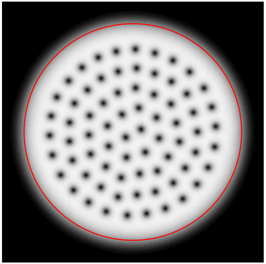

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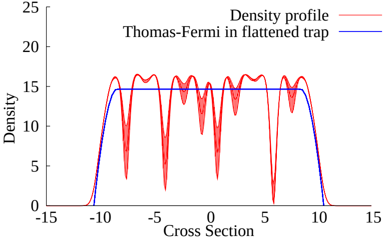

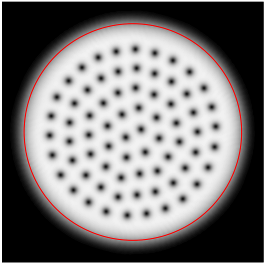

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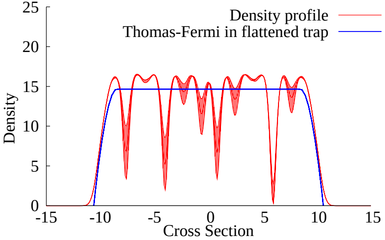

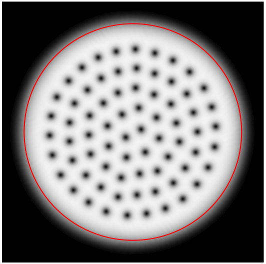

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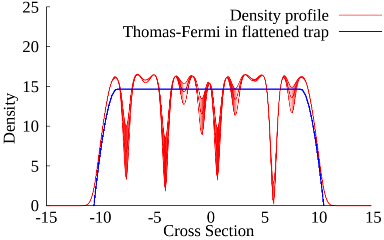

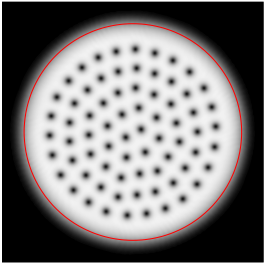

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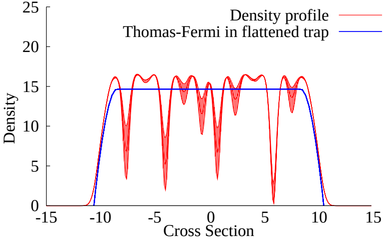

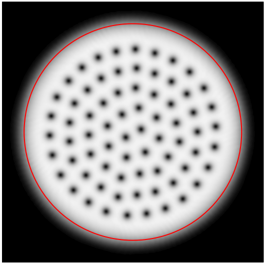

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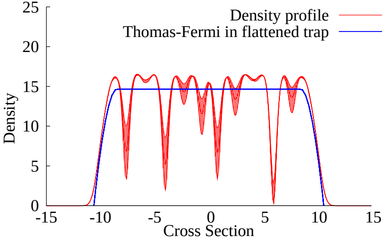

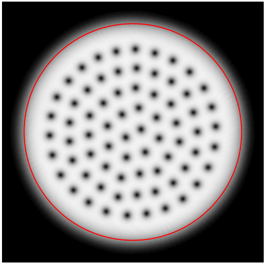

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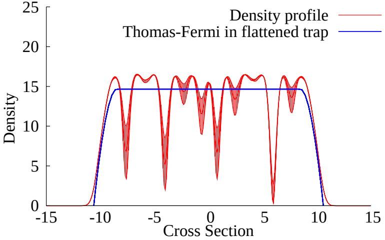

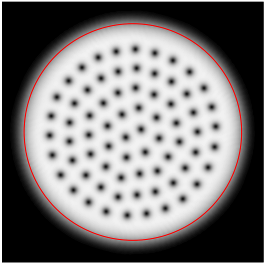

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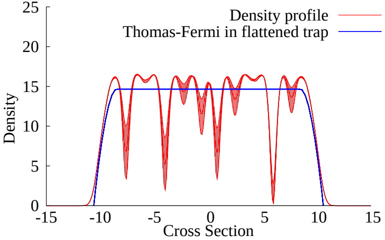

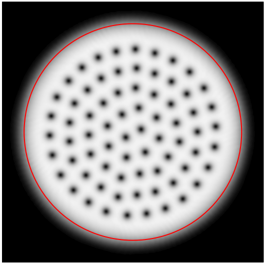

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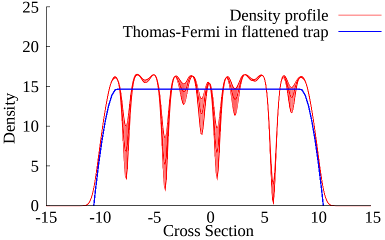

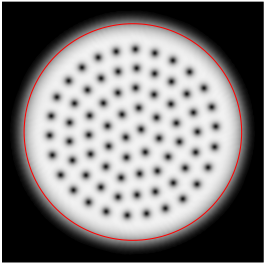

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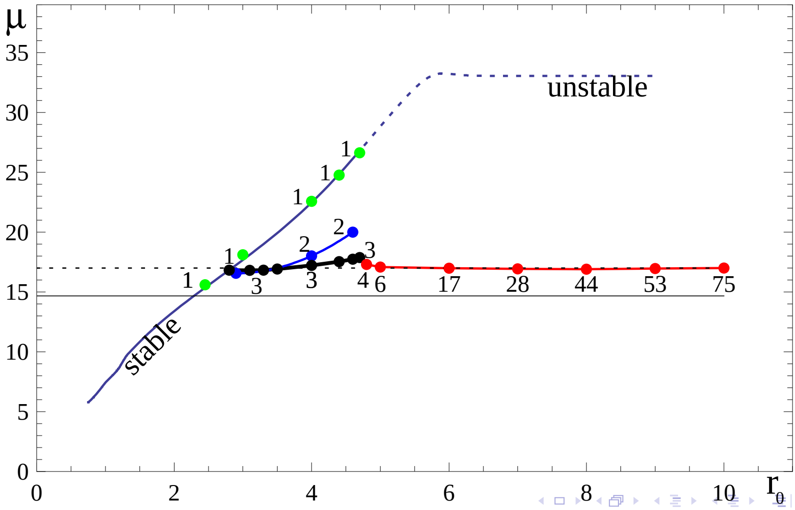
Why vortices: chemical potential vs size

$$\text{Thomas-Fermi}: \mu = f(r_0) \quad \text{Vortex}: \mu = \frac{U\gamma_{\text{net}}}{\Gamma}$$



Overview

1. Introduction to microcavity polaritons
2. Model and review of equilibrium results
   - Disorder-localised exciton model
   - Equilibrium mean field theory
3. Microscopic non-equilibrium model
   - Model and mean-field theory
   - Fluctuations
     - Stability of normal state — lasing vs condensation
     - Condensed spectrum
4. Macroscopic phenomenology
   - Gross Pitaevskii equation in an harmonic trap
   - Internal Josephson effect and spatial variation
     - Spin degree of freedom
     - Spin and spatial degrees of freedom
5. Conclusions

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- For weakly-interacting dilute Bose gas model:
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- Magnetic field: Δ
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Non-equilibrium spinor system

Spinor Gross-Pitaevskii equation:

$$i\partial_t \psi_L = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(r) + \frac{\Delta}{2} + U_0 |\psi_L|^2 + (U_0 - 2U_1)|\psi_R|^2 + i\left(\gamma_{\text{eff}}(\mu_B) - \kappa - \Gamma|\psi_L|^2\right) \right] \psi_L + J_1 \psi_R$$

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- Two-mode case (neglect spatial variation): [Wouters PRB '08]
- To recap results write $\psi_{L,R} = \sqrt{\rho_{L,R}} e^{i(\phi \pm \theta/2)}$, $\rho_{L,R} = R \pm z$.

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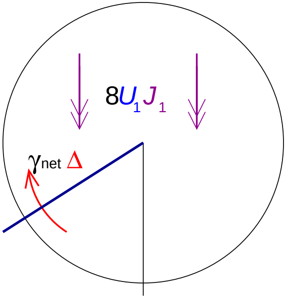

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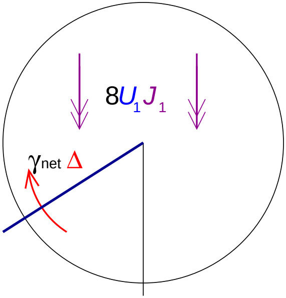

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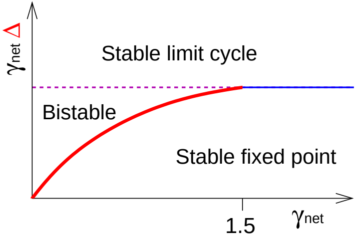

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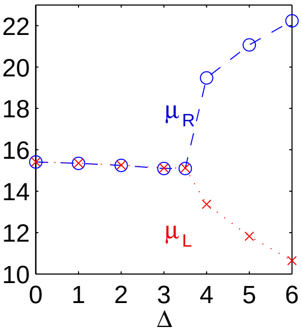

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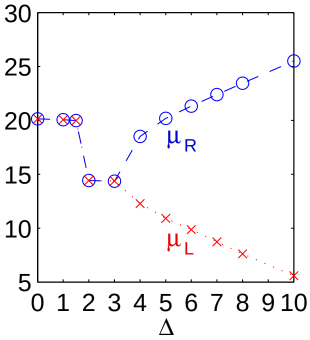
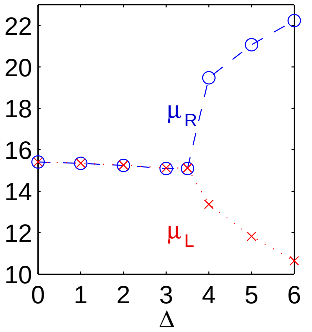
Δ causes $L(R)$ to grow (shrink)





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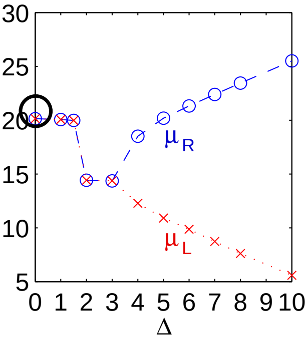

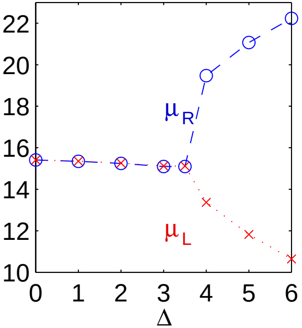

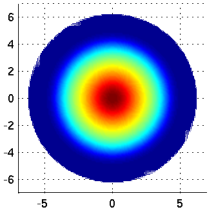

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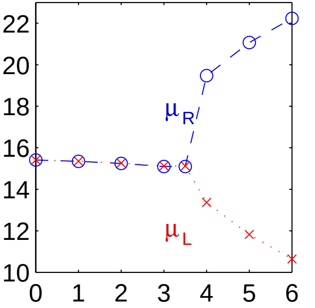

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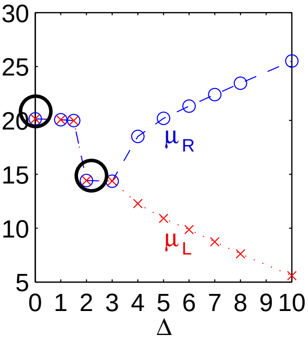
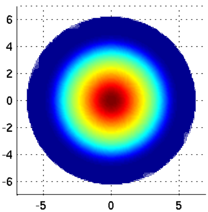
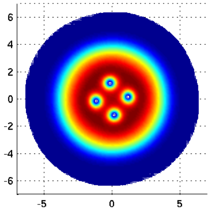
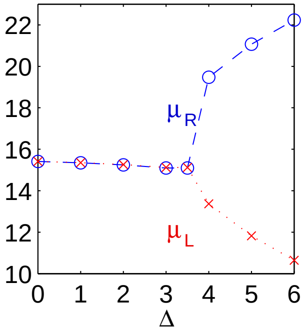






Trapped spinor system — phase portraits

"Simple" case not so simple



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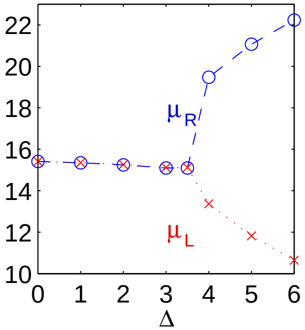

Examine phase portrait $\partial_t \theta$ vs θ

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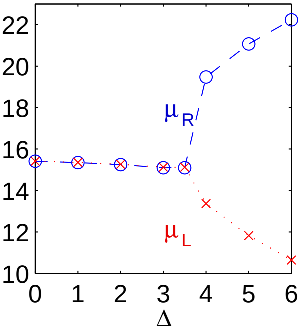
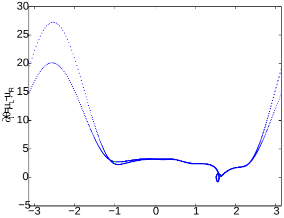
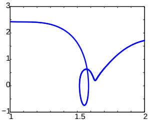




$\Delta = 3.20$



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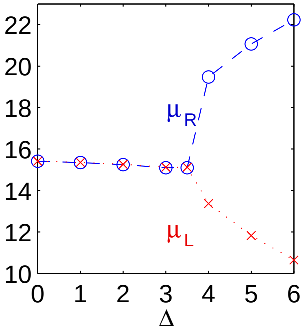

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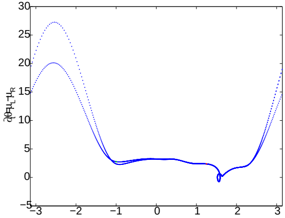

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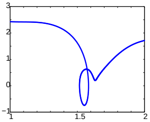
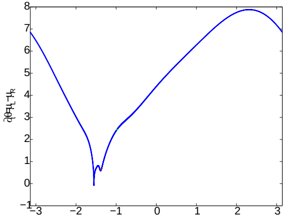
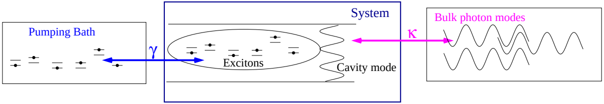


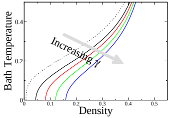


$\Delta = 3.25$

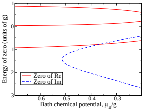
Conclusions

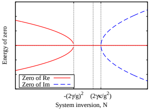
- Effects of pumping on mean-field theory



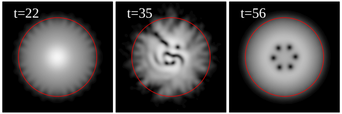


- Instability of normal state
  Translating: condensation ↔ lasing

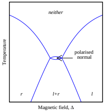


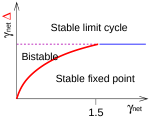


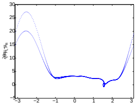
- Modification to Thomas-Fermi profile
  Spontaneous rotating vortex lattice



- Spinor model.
  Steady states & fluctuations.







Extra slides

Fluctuation corrections to phase boundary

Fluctuation corrections to density

$$\rho \rightarrow \rho_0 + \sum_{q} \langle \delta\psi^\dagger \delta\psi \rangle$$

In 2D system: modified critical condition:

$$\rho_s = \rho - \rho_{\text{normal}} = \# \frac{2m^2}{\hbar} k_B T$$

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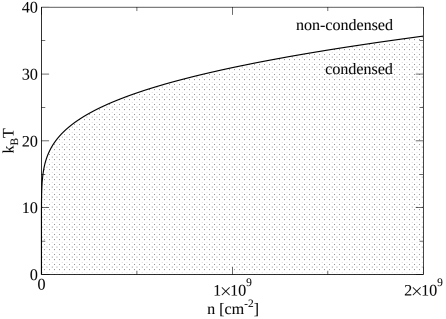

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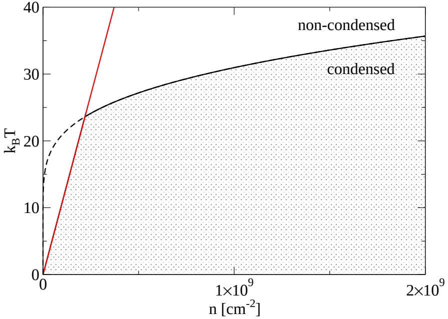

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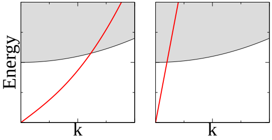

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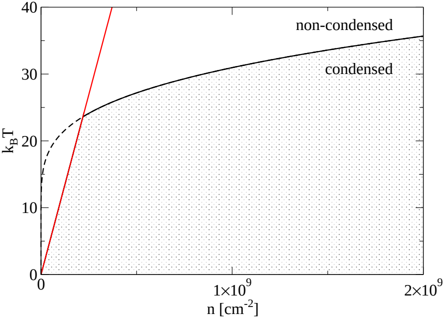

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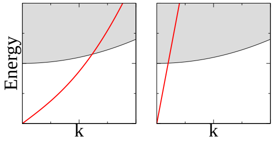

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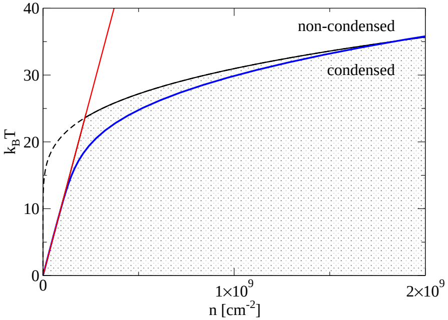

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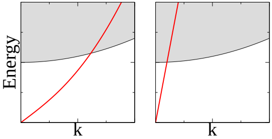

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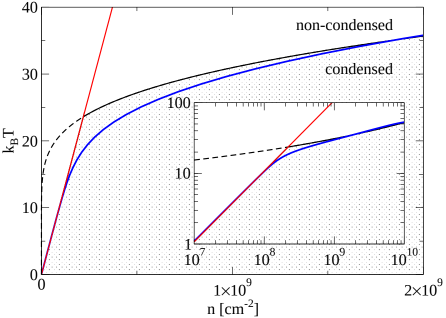

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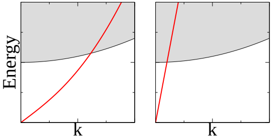

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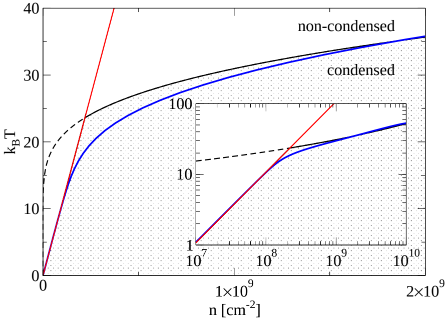
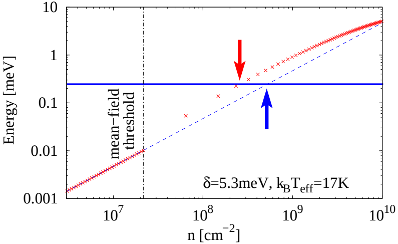


Second BCS crossover at $na_B^2 \simeq 1$

Blueshfit and experimental phase boundary

Blueshift:



Clean limit:

$$\delta E_{\mathrm{LP}} \simeq \mathcal{R}y_{\mathrm{X}} a_{\mathrm{X}}^2 n + \Omega_{\mathrm{R}} a_{\mathrm{X}}^2 n$$

Here: $\Omega_{\mathrm{R}} a_{\mathrm{X}}^2 \to \Omega_{\mathrm{R}} \xi^2$
[PRB 77 235313]

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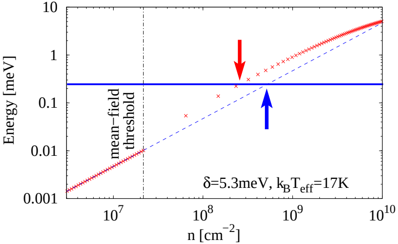

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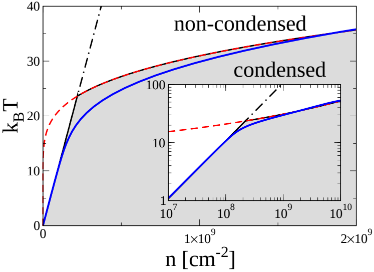
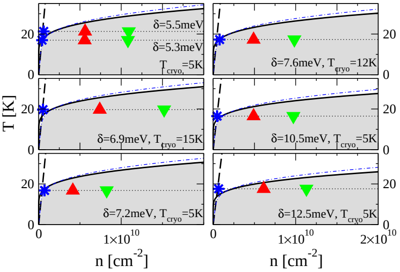
Phase diagram:

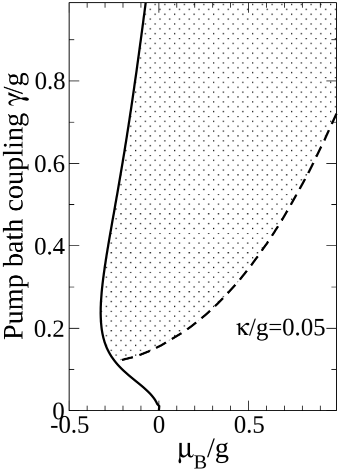


CdTe:



Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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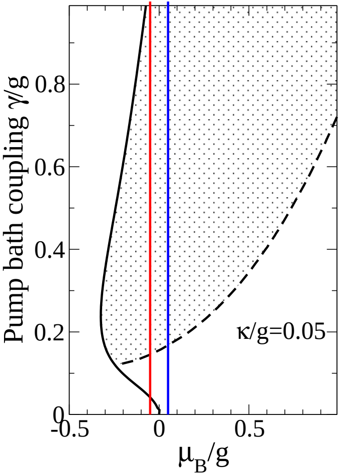

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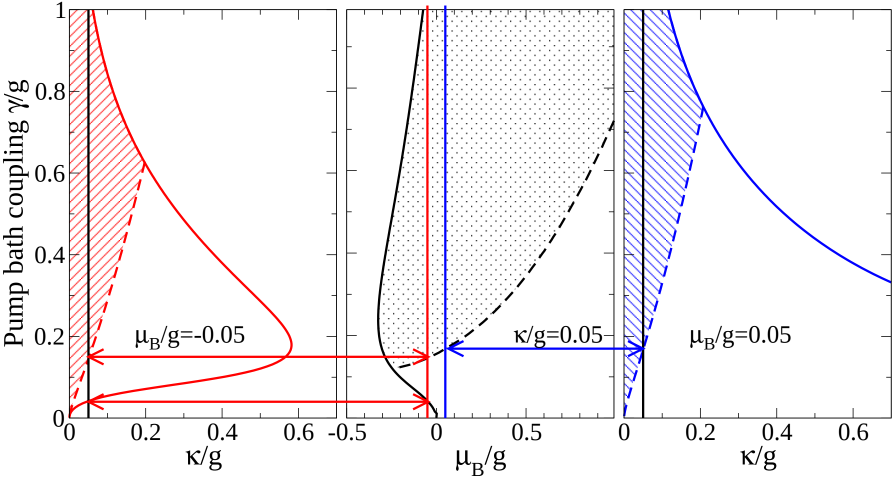

Finite size effects: Single mode vs many mode

$$\left\langle \psi^\dagger(r,t)\psi(0,r') \right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$$

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$\mathcal{D}_{\phi\phi}(t,r,r')$ from sum of phase modes. Study $ct \gg r$ limit:

$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_n^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{|(\omega+ix)^2 + x^2 - (n\Delta)^2|^2}$$

Finite size effects: Single mode vs many mode

$$\left\langle \psi^\dagger(r,t)\psi(0,r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$$

$\mathcal{D}_{\phi\phi}(t,r,r')$ from sum of phase modes. Study $ct \gg r$ limit:

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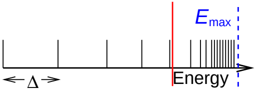
$\Delta \ll \sqrt{x/t} \ll E_{\text{max}}$



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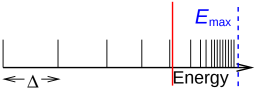

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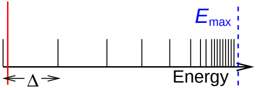

$\mathcal{D}_{\phi\phi} \sim \left(\frac{\pi C}{2x}\right)\left(\frac{t}{2x}\right)$

Relating finite-size spectrum to self phase modulation

Single mode spectrum:

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Connection to Kubo Formula [Eastham and Whittaker, arXiv:0811.4333]

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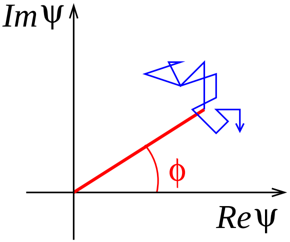

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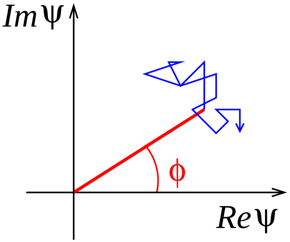

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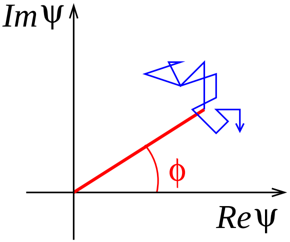

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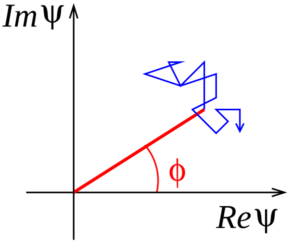

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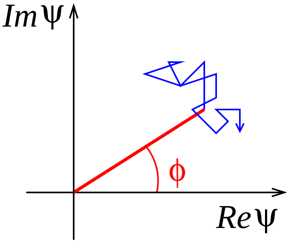

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&= \frac{CU^2}{2\Gamma^3}\left[\Gamma t - 1 + e^{-\Gamma t}\right]
\end{aligned}$$

Instability of Thomas-Fermi: details

$\psi = \sqrt{\rho}e^{i\phi} \quad \mathbf{v} = \frac{\hbar}{m}\nabla\phi$

$$\frac{1}{2}\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = (\gamma_{\text{net}} - \Gamma\rho)\rho$$

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$$\delta\rho_{n,m}(r, \theta, t) = e^{im\theta} h_{n,m}(r) e^{i\omega_{n,m} t}$$

$$\omega_{n,m} = \omega 2\sqrt{m(1 + 2n) + 2n(n + 1)}$$

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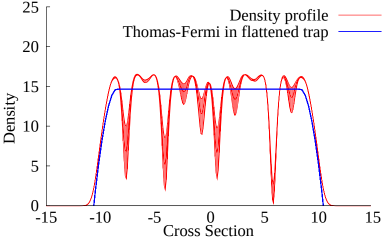
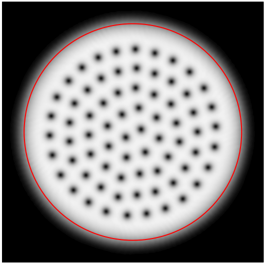
Add weak pumping/decay:

$$\omega_{n,n} \to \omega_{n,m} + i\gamma_{\text{net}}\left[\frac{m(1+2n) + 2n(n+1) - m^2}{2m(1+2n) + 4n(n+1) + m^2}\right]$$

Instability

Why vortices





$$\nabla\cdot[\rho(\mathbf{v}-\Omega\times\mathbf{r})] = (\gamma_{\text{net}}\Theta(r_0-r)-\Gamma\rho)\rho$$

$$\mu = \frac{m}{2}|\mathbf{v}-\Omega\times\mathbf{r}|^2 + \frac{m}{2}r^2(\omega^2-\Omega^2) + U\rho - \frac{\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}$$

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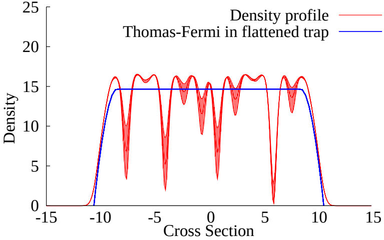

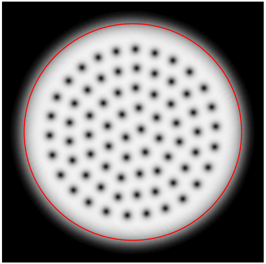

Rotating solution: $i\partial_t\psi = (\mu - 2\Omega L_z)\psi$

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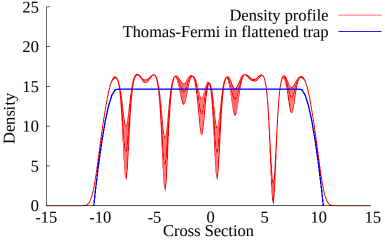

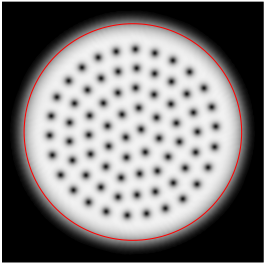

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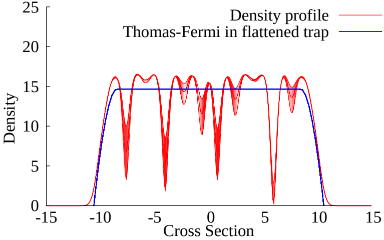

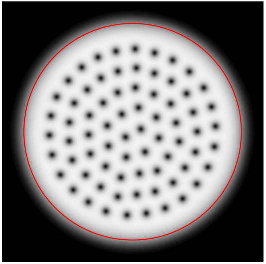

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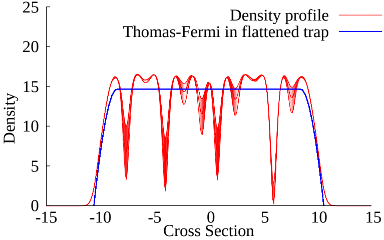

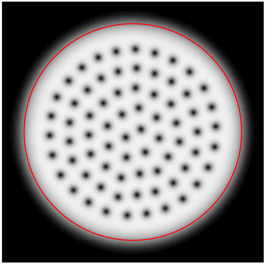

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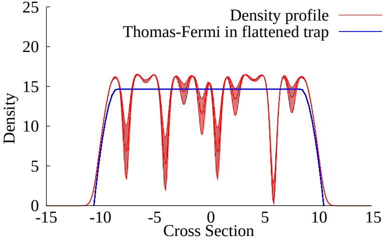

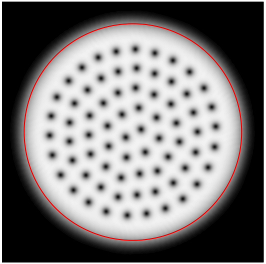

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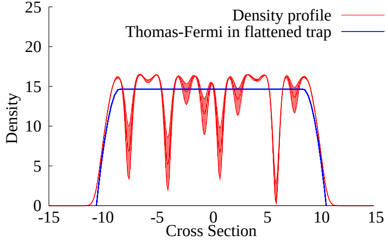

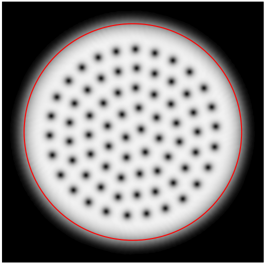

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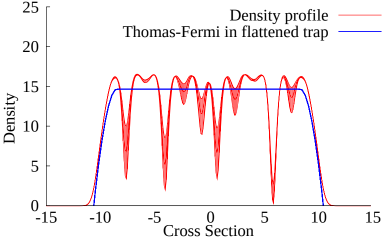

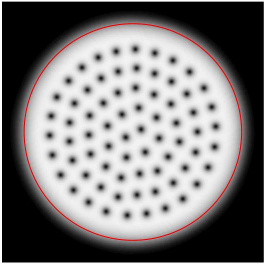

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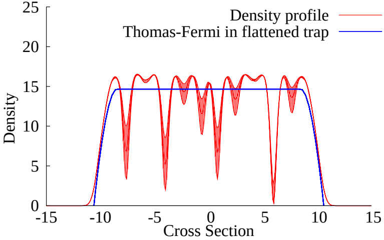

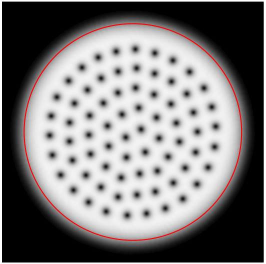

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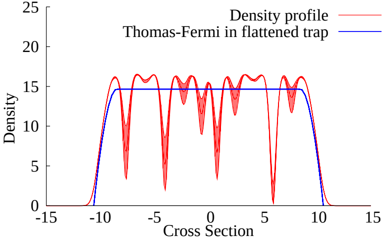

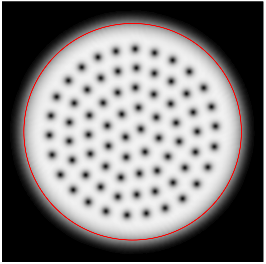

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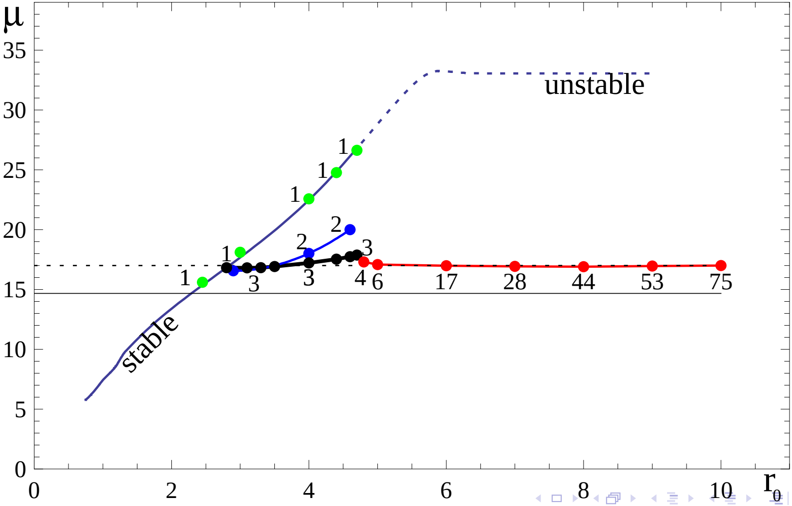
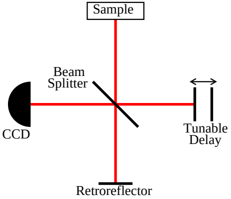
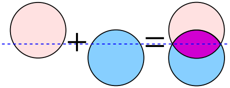
Why vortices: chemical potential vs size

$$\text{Thomas-Fermi} : \mu = f(r_0) \quad \text{Vortex} : \mu = \frac{U\gamma_{\text{net}}}{\Gamma}$$

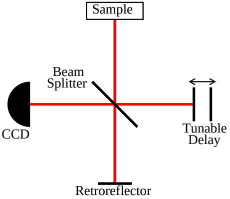


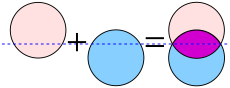
Observing vortices: fringe pattern

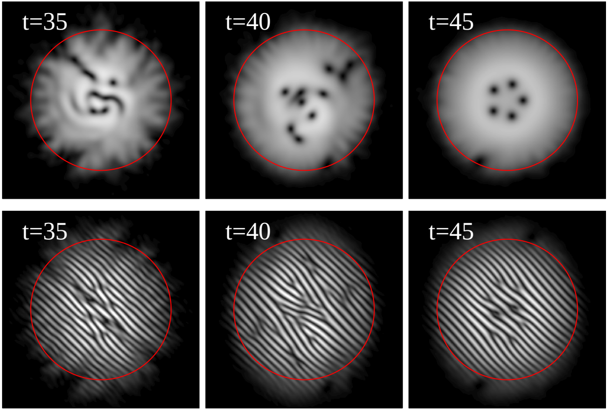




Observing vortices: fringe pattern







Spin in terms of ~~two~~four-level systems

To include spin, replace 2 level system with 4 levels: $|0\rangle, |L\rangle, |R|\rangle, |LR\rangle$

- Bi-exciton binding $E_{XX} \leftrightarrow U_1$
- Mean-field: find polarisation given ψ_L, ψ_R.
- E_{XX} has weak effect on T_c

[Marchetti *et al* PRB, '08]

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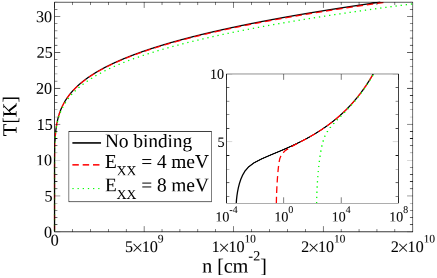
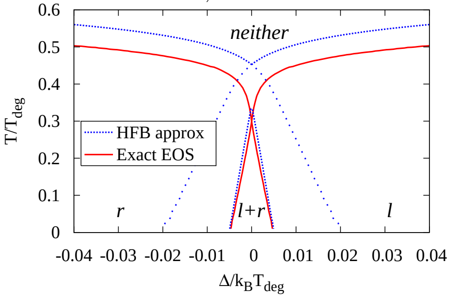


[Marchetti *et al* PRB, '08]

Equilibrium phase diagrams

$J_1 = J_2 = 0$.

For $U_1 = 0.5$, $\Psi_{L,R}$ decouple.



Circular $\longrightarrow$ Eliptical transitions.

[Rubo *et al* Phys. Lett. A '06; Keeling, Phys. Rev. B '08]

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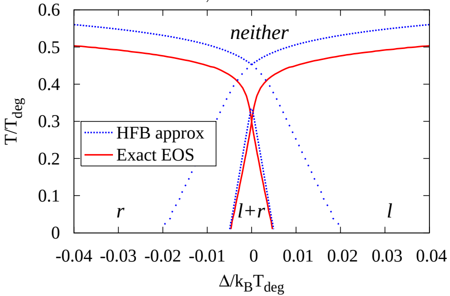

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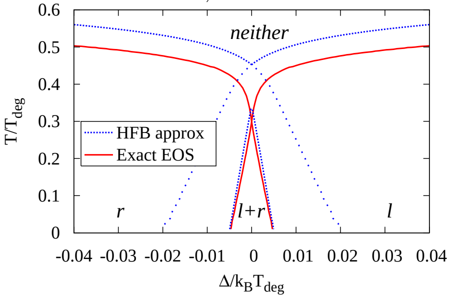

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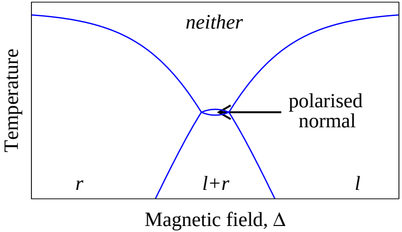

Separate Ising/XY transitions.

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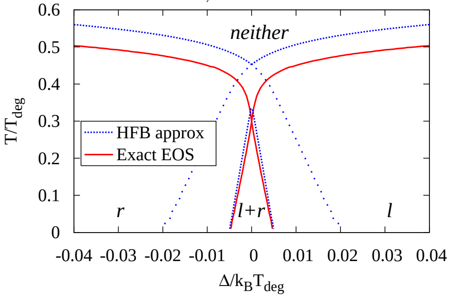

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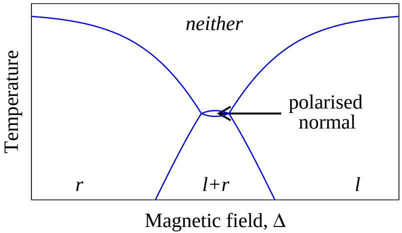

Separate Ising/XY transitions.

$J_1 \neq 0$: Eqbm state locked.

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Mathematical outline

- 2D Single component equation of state: $n(\mu, T) = Tf(x = \mu/T)$
- For two components:

$$n_0 = T\left[f\left(\frac{\mu+\Omega}{T}\right) + f\left(\frac{\mu-\Omega}{T}\right)\right]$$

- At critical point for one component:

$$n_0 = T\left[f_c + f\left(x_c + \frac{2\Omega}{T}\right)\right]$$

- Hence:

$$T = \frac{n_0}{f_c + f\left(x_c + \frac{2\Omega}{T}\right)}$$

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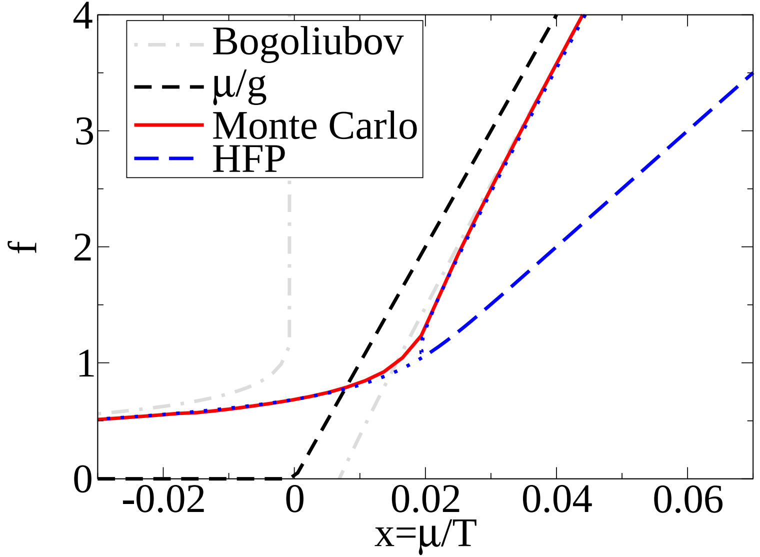

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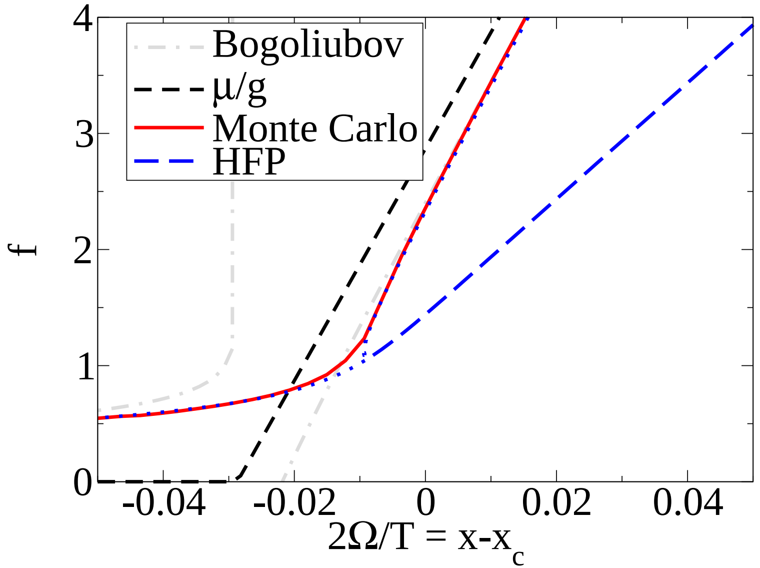

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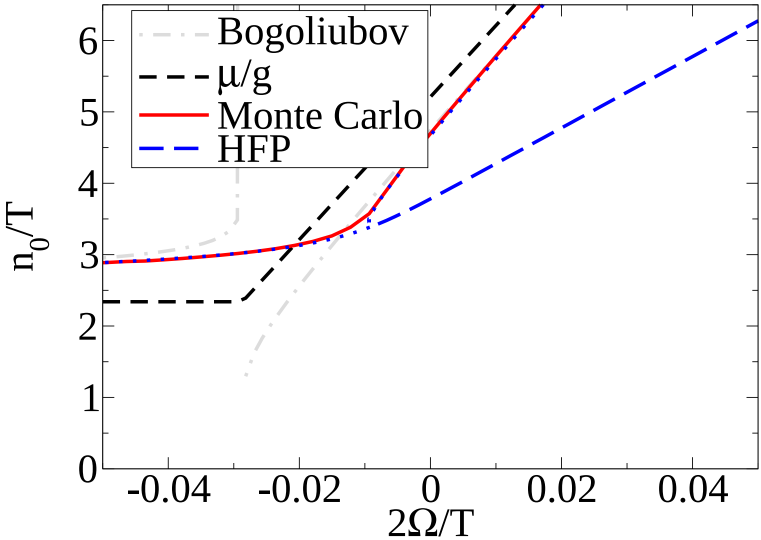

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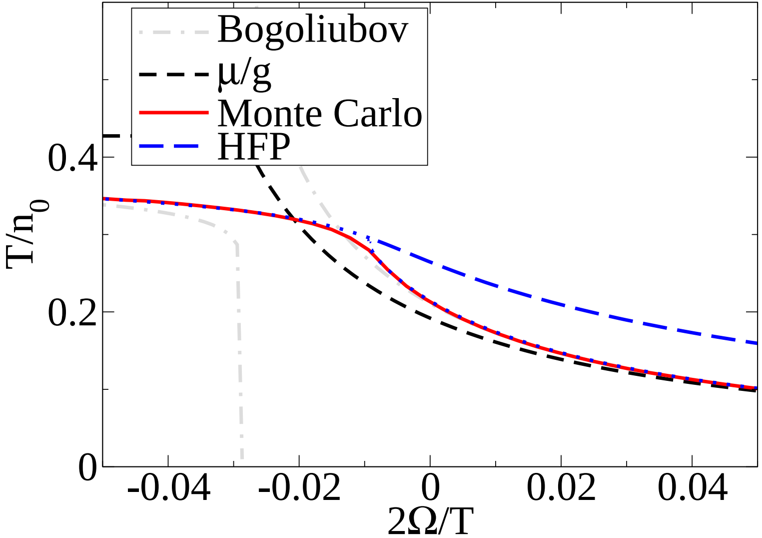

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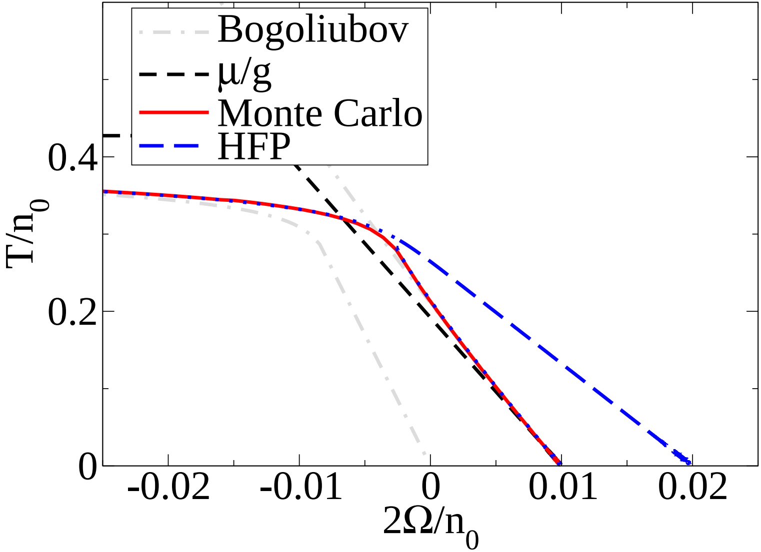

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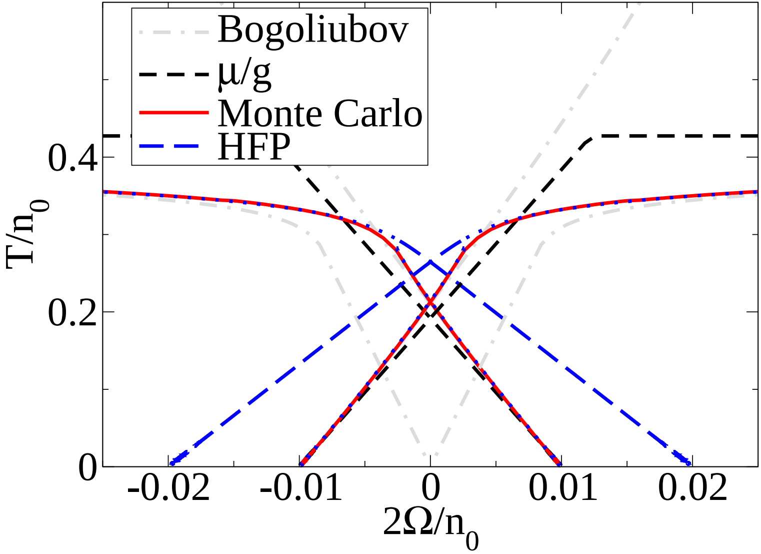
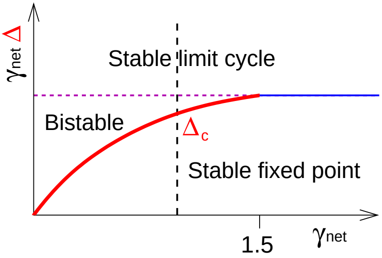


Two-mode model bistability



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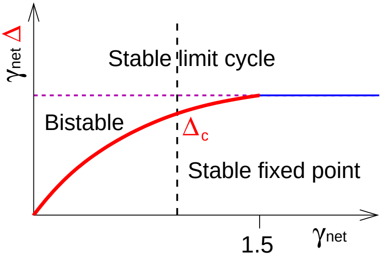

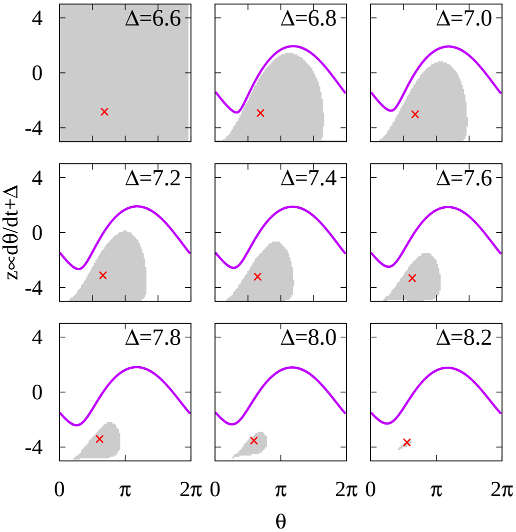

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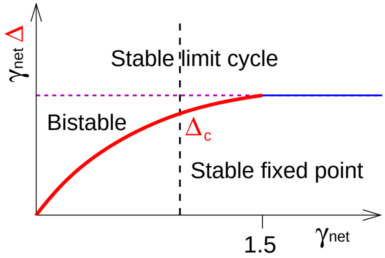
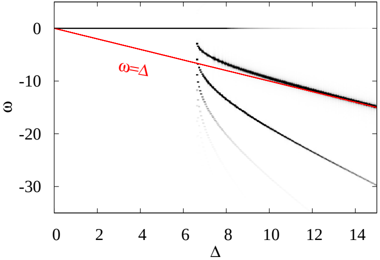
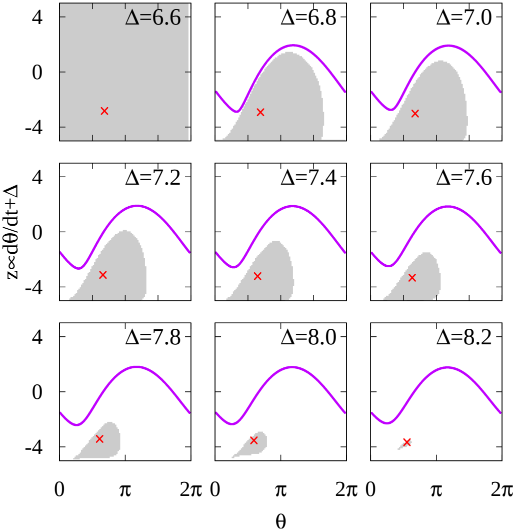




$\Delta \gtrsim \Delta_c$: $\omega \simeq [\ln(\Delta - \Delta_c)]^{-1}$



Spatial freedom: Homgeneous case $\Delta < \Delta_c$

$$\ddot{\theta} + 2\gamma_{\text{net}}\dot{\theta} = 8 U_1 J_1 R_0 \sin(\theta) - 2\gamma_{\text{net}}\Delta$$

- Steady state condition: $8 U_1 J_1 R_0 \sin(\theta) = 2\gamma_{\text{net}}\Delta$
- $\psi_{L,R} \to e^{-i\mu t}\left(\psi^0_{L,R} + u_1 e^{-ik\cdot r + (-i\omega-\kappa)t} + v_1^* e^{ik\cdot r + (i\omega - \kappa)t}\right)$
- Define $\Omega_p^2 = -8 U_1 J_1 R_0 \cos(\theta)$. At $k = 0$

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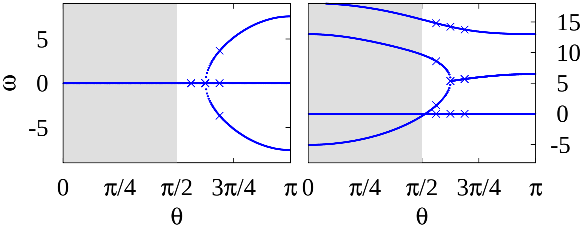
$$\omega - i\kappa = 0, -2i\gamma_{\text{net}}$$

$$-i\gamma_{\text{net}} \pm i\sqrt{\gamma_{\text{net}} - \Omega_p^2}$$



Stability requires $\Omega_p^2 > 0$. If $\Omega_p^2 < \gamma_{\text{net}}$ overdamped.

Spatial variation

Varieties of behaviour possible as $\theta(\mathbf{r})$, not $\bar{\theta}$ needed to define state.

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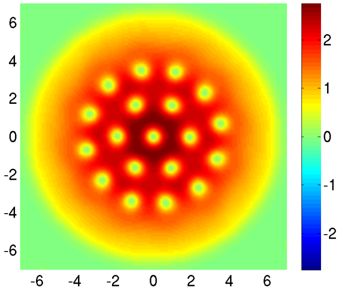

$J_1 = 0.5$; $r_0 > r_{TF}$; $\Delta = 6$

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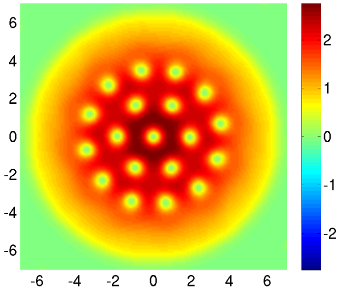

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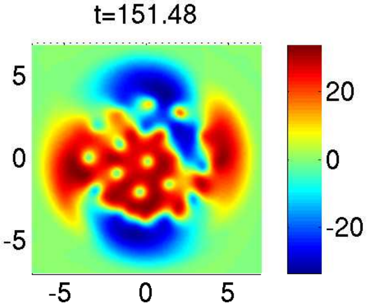

$J_1 = 1; r_0 > r_{TF}; \Delta = 6$
Counter-rotating.

Superfluidity

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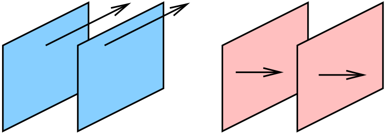

$$\begin{aligned}\chi_{ij}(\omega = 0, \mathbf{q} \to 0) &= \langle J_i(\mathbf{q}) J_j(\mathbf{q})\rangle \\ &= \chi_T(q)\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) + \chi_L(q)\frac{q_i q_j}{q^2}\end{aligned}$$

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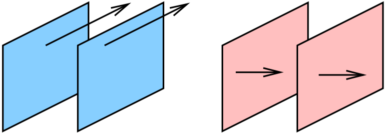

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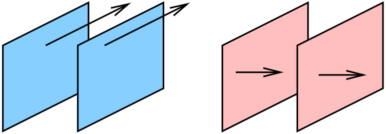

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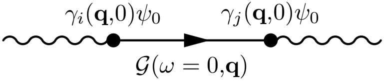
$\Delta\chi_{ij}(q) =$  $+ \ldots$

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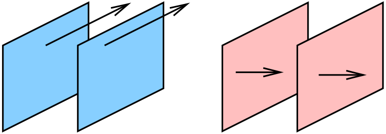

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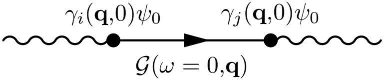
$\Delta\chi_{ij}(q) =$  $+ \dots$

$$= \gamma_i(q) \mathcal{G}_R(\omega = 0, \vec{q}) \gamma_j(q) + \dots$$

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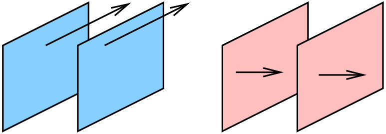
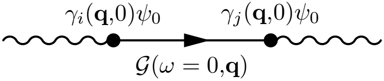
$$\mathbf{J} = \rho \mathbf{v} = \Psi^\dagger i\hbar\nabla\Psi = |\Psi|^2\nabla\phi$$



$$\chi_{ij}(\omega = 0, \mathbf{q} \to 0) = \langle J_i(\mathbf{q}) J_j(\mathbf{q}) \rangle$$
$$= \chi_T(q)\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right) + \chi_L(q)\frac{q_i q_j}{q^2}$$

Superfluid part,
$\rho_s \propto \lim_{q\to 0}(\chi_L - \chi_T)$.
$J_i(q) = \langle \psi^\dagger \gamma_i(q) \psi \rangle$

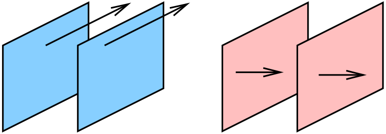
$\Delta\chi_{ij}(q) =$  $+ \ldots$

$$= \gamma_i(q)\mathcal{G}_R(\omega = 0, \vec{q})\gamma_j(q) + \ldots$$
$$\propto \frac{q_i(\ldots)q_j}{(\omega + ix)^2 + x^2 - c^2\mathbf{q}^2} + \ldots$$

Superfluidity

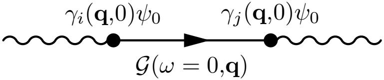
Superfluidity:

$$\mathbf{J} = \rho \mathbf{v} = \Psi^\dagger i\hbar\nabla\Psi = |\Psi|^2 \nabla\phi$$



$$\chi_{ij}(\omega = 0, \mathbf{q} \to 0) = \langle J_i(\mathbf{q}) J_j(\mathbf{q}) \rangle$$
$$= \chi_T(q) \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right) + \chi_L(q) \frac{q_i q_j}{q^2}$$

Superfluid part,
$\rho_s \propto \lim_{q \to 0} (\chi_L - \chi_T)$.
$J_i(q) = \langle \psi^\dagger \gamma_i(q) \psi \rangle$
Static ρ_S survies

$\Delta\chi_{ij}(q) =$  $+ \ldots$

$$= \gamma_i(q) \mathcal{G}_R(\omega = 0, \vec{q}) \gamma_j(q) + \ldots$$
$$\propto \frac{q_i \,(\ldots)\, q_j}{(\omega + ix)^2 + x^2 - c^2 \mathbf{q}^2} + \ldots$$