Comparing non-equilibrium polariton condensation and lasing

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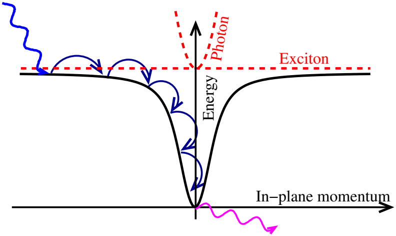
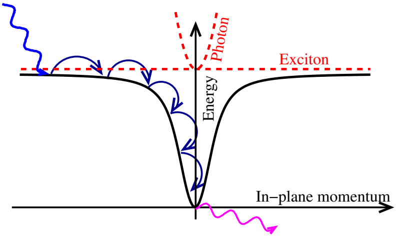
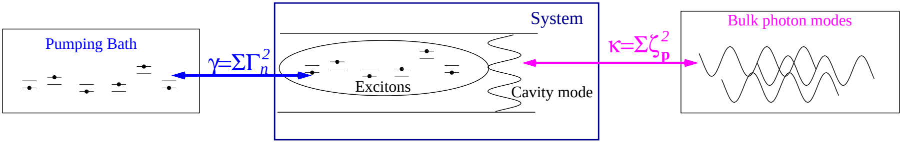
Aims



Aims

- Non-equilibrium diagram (Keldysh) approach.
- Modelling system with particle flux.
- Show connection between:
  - Equilibrium BEC: chemical potential $\mu \rightarrow 0$.
  - Simple LASER: round-trip gain exceeds loss.





Overview

1. Non-equilibrium Green's functions
   - Definitions of Green's function
   - Diagrammatic approach
   - Example: decaying photons
2. Polariton condensation and lasing
   - Self energy for the Dicke model
   - Analytic properties of Green's functions
   - Normal state instability
3. Summary

Need for multiple Green's functions

- Consider definitions:

$$iD^{<}(t,r) = \langle \psi^\dagger(0,0)\psi(t,r)\rangle, \qquad iD^{>}(t,r) = \langle \psi(t,r)\psi^\dagger(0,0)\rangle,$$

- Fourier transform:

$$iD^{<}(\omega,k) = n_B(\omega)\rho(\omega,k), \qquad iD^{>}(\omega,k) = [n_B(\omega)+1]\rho(\omega,k).$$

- In equilibrium, $n_B(\omega) = \dfrac{1}{e^{\beta\omega}-1}$
- Out of equilibrium, need (at least) two Green's functions.

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Retarded and Keldysh Green's functions

- In fact, consider:

$$D^R(t,r) = -i\theta(t)\left\langle[\psi(t,r),\psi^\dagger(0,0)]\right\rangle, \quad D^A(t,r) = [D^R(-t,-r)]^*$$

$$D^K(t,r) = -i\left\langle[\psi(t,r),\psi^\dagger(0,0)]_+\right\rangle.$$

- For free fields:
  - $\psi(t,r) = \sum_k \psi_k e^{-i\omega_k t - ik\cdot r}$.
  - Hence:

$$D^R(t,r) = -i\theta(t)\sum_k e^{-i\omega_k t - ik\cdot r}, \qquad D^R(\omega,k) = \frac{1}{\omega - \omega_k + i0},$$

$$D^K(\omega,k) = -(2\pi i)\delta(\omega-\omega_k)(2n_B(\omega_k)+1).$$

  - Density of states:

$$D^R(\omega,k) - D^A(\omega,k) = -(2\pi i)\delta(\omega-\omega_k).$$

  - Can rewrite Keldysh Green's function:

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Constructing diagrammatic expansion

- Need Heisenberg picture $\psi(t)$:

$$\psi(t) = e^{iHt}\psi e^{-iHt} = U^{-1}(t)\tilde{\psi}(t)U(t)$$

Interaction picture: $H = H_0 + H_{\text{int}}$ $\qquad \tilde{\psi}(t) = e^{iH_0 t}\psi e^{-iH_0 t}$,

- Gives equation for $U(t)$:

$$\partial_t U = -ie^{iH_0 t}H_{\text{int}}e^{-iH_0 t}U, \qquad U(t) = T\exp\left[-i\int^t \tilde{H}_{\text{int}}(t')dt'\right].$$

- Aim: Green's function as time-ordered expectation

$$\langle T[A(t)B(t')]\rangle = \begin{cases} \langle A(t)B(t')\rangle & t > t' \\ \langle B(t')A(t)\rangle & t < t' \end{cases}$$

- In time ordered product: $D = \langle T\left[A(t)B(t')\right]\rangle = \left\langle T\left[\tilde{A}(t)\tilde{B}(t')U\right]\right\rangle$

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Keldysh contour

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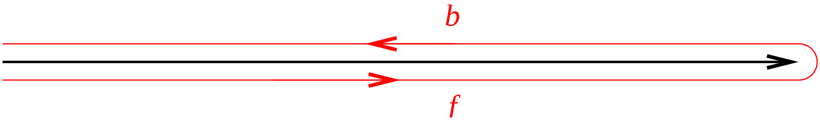

Ordering T_c: $(t, b) > (t', f)$ for all t, t'.

- Hence: $D^>(t) = -i\langle\psi(t)\psi^\dagger(0)\rangle = -i\left\langle T_c\left(\psi(t,b)\psi^\dagger(0,f)\right)\right\rangle$

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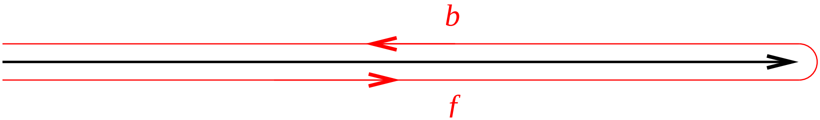

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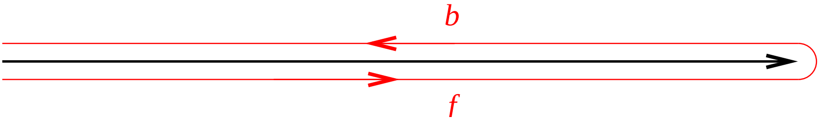
Keldysh contour and Green's functions



- Define $\psi_\pm(t) = \frac{1}{\sqrt{2}}\left(\psi(t,f) \pm \psi(t,b)\right),$
- Gives matrix of Green's functions:

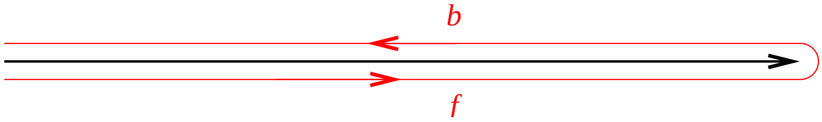
$$D = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = -i\left\langle T_c \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \begin{pmatrix} \psi_+^\dagger(0) & \psi_-^\dagger(0) \end{pmatrix} \right\rangle$$

- For example:

$$D^R = -\frac{i}{2}\left\langle T_c \left[\left(\psi(t,f) + \psi(t,b)\right)\left(\psi^\dagger(0,f) - \psi^\dagger(0,b)\right)\right]\right\rangle$$

$$= -\frac{i}{2}\left\langle \psi(t)\psi^\dagger(0) - \psi^\dagger(0)\psi(t) + \begin{cases} \psi(t)\psi^\dagger(0) - \psi^\dagger(0)\psi(t) & t>0 \\ \psi^\dagger(0)\psi(t) - \psi(t)\psi^\dagger(0) & t<0 \end{cases}\right\rangle$$

Keldysh contour and Green's functions



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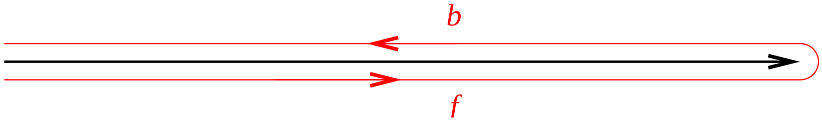

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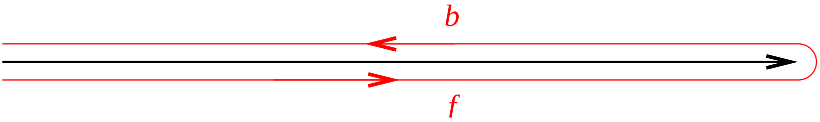
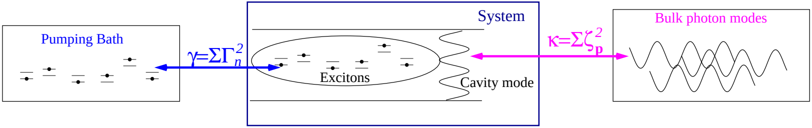
Non-equilibrium diagrammatic theory: Summary



- $\begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = -i \left\langle T_c \left[\begin{pmatrix} \tilde{\psi}_+(t) \\ \tilde{\psi}_-(t) \end{pmatrix} \begin{pmatrix} \tilde{\psi}^\dagger_+(0) & \tilde{\psi}^\dagger_-(0) \end{pmatrix} U \right] \right\rangle$
- Propagator:

$$U = \exp\left(-i \int_C \tilde{H}_{\text{int}}(t) dt\right)$$
$$= \exp\left(-i \int_{-\infty}^{\infty} \left[\tilde{H}_{\text{int}}(t, f) - \tilde{H}_{\text{int}}(t, b)\right] dt\right).$$

Illustration: coupling to a decay bath



- Coupling: $H_{int} = \sum_{k,P} \zeta_{k,P} \left(\psi_k^\dagger \Psi_P + \Psi_P^\dagger \psi_k \right)$
- To find propagator need:

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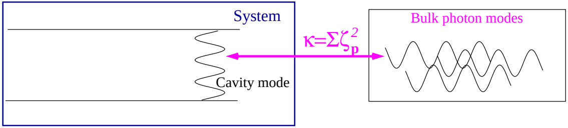

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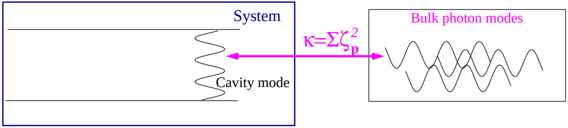

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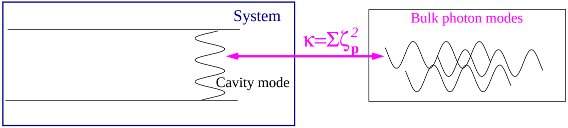

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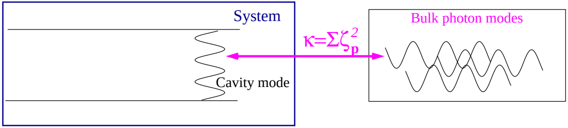
Illustration: coupling to a decay bath



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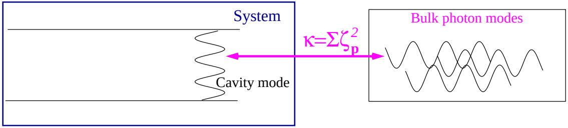

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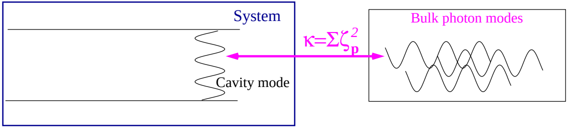

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$$D^{-1} = D_0^{-1} - \Sigma = \begin{pmatrix} 0 & \omega - \omega_k - i0 \\ \omega - \omega_k + i0 & i0 \ldots \end{pmatrix} - \begin{pmatrix} 0 & \Sigma^R \\ \Sigma^A & \Sigma^K \end{pmatrix}.$$

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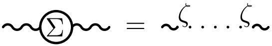

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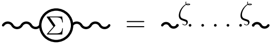
- e.g, Retarded self energy 

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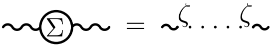
- e.g, Retarded self energy 

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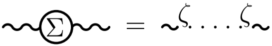
- e.g, Retarded self energy 

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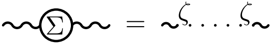
- e.g, Retarded self energy 

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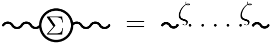
- e.g, Retarded self energy 

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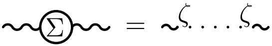
- e.g, Retarded self energy 

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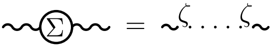
- e.g, Retarded self energy 

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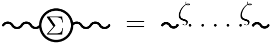
- e.g, Retarded self energy 

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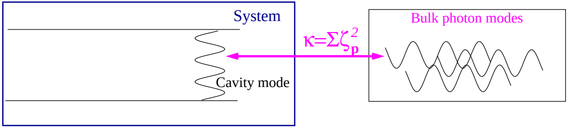

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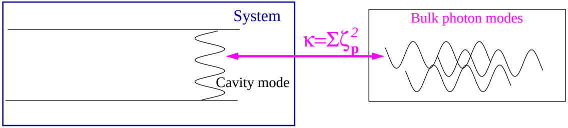

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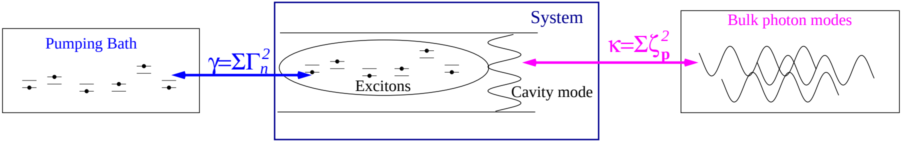
Overview

Two-level system model of polaritons



- Two-level system Hamiltonian

$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i \left(b_i^\dagger b_i - a_i^\dagger a_i \right) + \sum_{i,k} g \left(\psi_k^\dagger a_i^\dagger b_i + \text{H.c.} \right).$$

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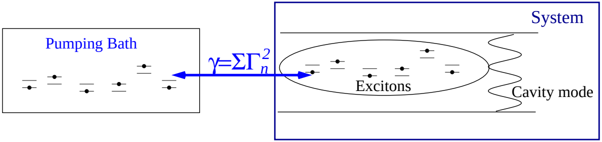

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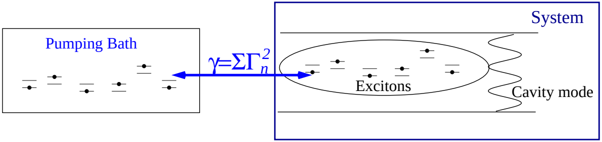

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- Treat sites as two-level system (exciton/no-exciton)
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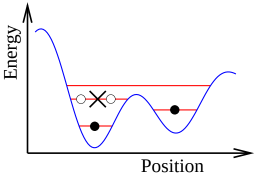

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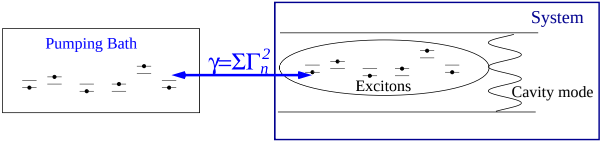

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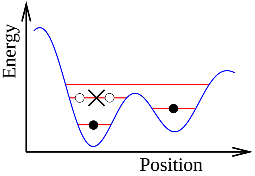

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Polariton self energy

- Include repeated exciton ↔ photon interconversion use

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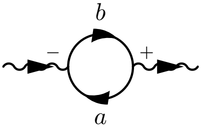
- Retarded self energy $\Sigma^{-+}_{\psi^\dagger\psi} =$ 

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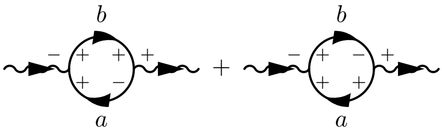
- Retarded self energy $\Sigma^{-+}_{\psi^\dagger\psi} =$ 

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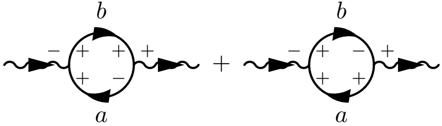
- Retarded self energy $\Sigma^{-+}_{\psi^\dagger\psi} =$ 

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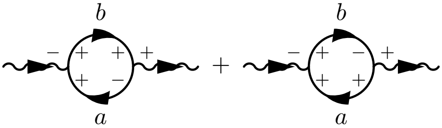
- Retarded self energy $\Sigma^{-+}_{\psi^\dagger\psi} =$



- Using fermion Green's functions G:

$$\Sigma^{-+}_{\psi^\dagger\psi} = -i\left(\frac{g}{\sqrt{2}}\right)^2 \int \frac{d\nu}{2\pi} \sum_i \Big[G^A_{a^\dagger a}(\nu) G^K_{b^\dagger b}(\nu+\omega) + G^K_{a^\dagger a}(\nu) G^R_{b^\dagger b}(\nu+\omega) \Big].$$

- Fermions coupled to pumping bath:

$$G^R_{b^\dagger b, a^\dagger a} = \frac{1}{\nu \mp \epsilon_i + i\gamma}, \qquad G^K_{b^\dagger b, a^\dagger a} = -\frac{2i\gamma \tanh\left[\frac{\beta}{2}(\nu \pm \mu_B)\right]}{(\nu \mp \epsilon_i)^2 + \gamma^2},$$

- β, μ_B describe pumping bath.

Inverse polariton Green's function

$$\left[D^R(\omega,k)\right]^{-1} = \omega - \omega_k + i\kappa$$
$$+ g^2 n\gamma \int \frac{d\nu}{2\pi} \frac{\tanh\left[\frac{\beta}{2}(\nu+\omega-\mu_B)\right]}{\left[(\nu+\omega-\epsilon_i)^2+\gamma^2\right](\nu+\epsilon_i-i\gamma)}$$
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Meaning of poles of Green's functions

- Write: $\left[D^R(\omega)\right]^{-1} = A(\omega) + iB(\omega) \qquad \left[D^{-1}(\omega)\right]^K = iC(\omega),$
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- Physical Luminescence:
  $iD^<(\omega) = \frac{i}{2}\left[D^K(\omega) - \left(D^R(\omega) - D^A(\omega)\right)\right] = \frac{C(\omega) - 2B(\omega)}{2[A(\omega)^2 + B(\omega)^2]}.$
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- For decaying photons:

$A(\omega) = \omega - \omega_k$	Describes location of normal modes
$B(\omega) = \kappa$	Describes linewidths of these modes
$C(\omega) = 2\kappa(2n_B(\omega) + 1)$	Describes their occupation

- With coupling to pumped excitons $B(\omega)$ can vary (and vanish) $\rightarrow$ effective chemical potential.

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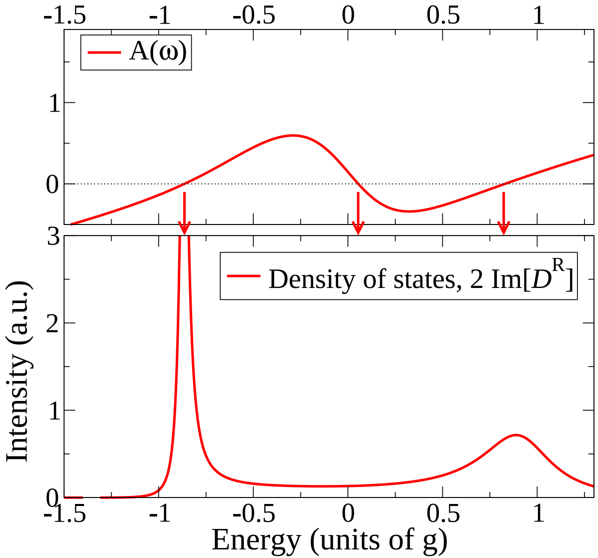
Inverse polariton Green's functions



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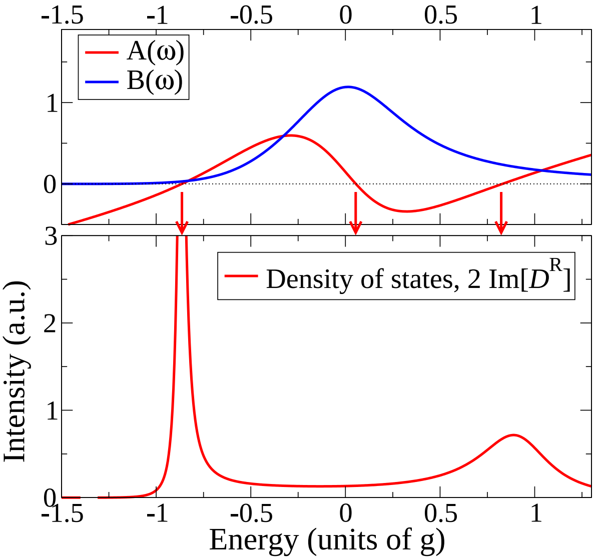

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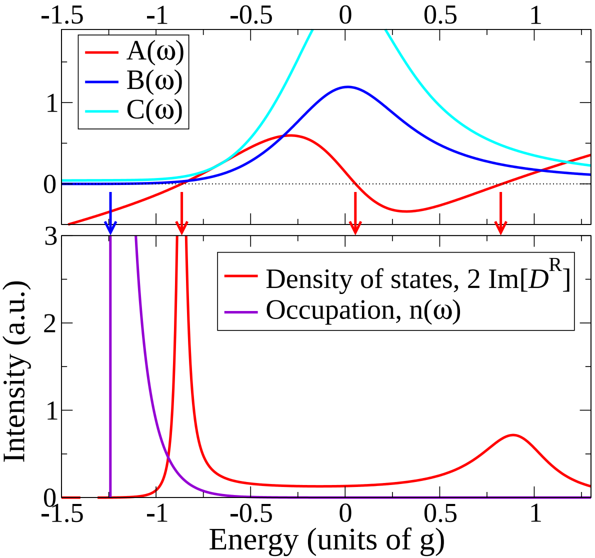

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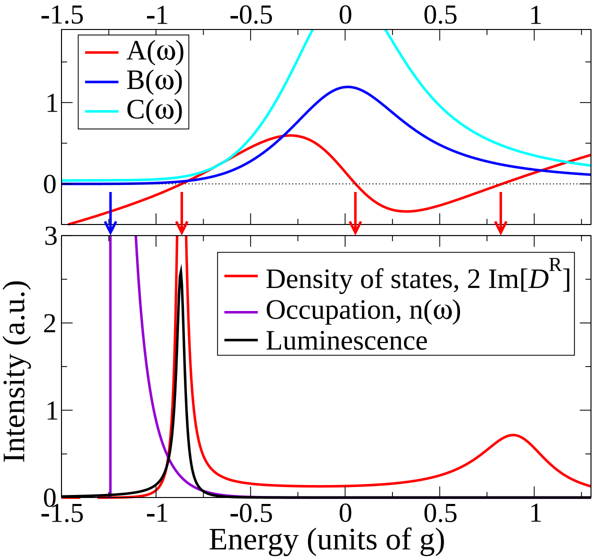

Evolution with pumping

From before:

$$B(\omega) = \kappa + g^2 n\gamma^2 \int \frac{d\nu}{2\pi} \frac{\tanh\left[\frac{\beta}{2}(\nu+\omega-\mu_B)\right] - \tanh\left[\frac{\beta}{2}(\nu+\mu_B)\right]}{\left[(\nu+\omega-\epsilon_i)^2+\gamma^2\right]\left[(\nu+\epsilon_i)^2+\gamma^2\right]},$$

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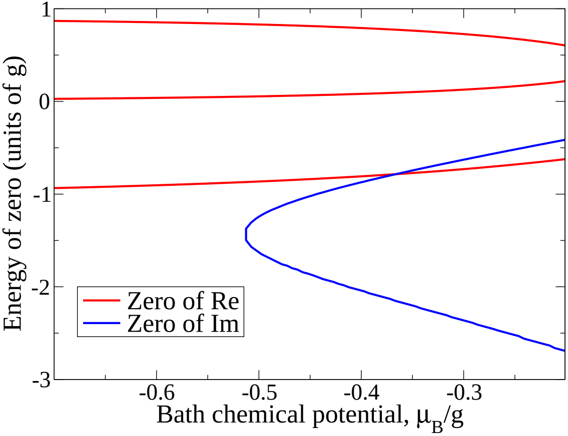
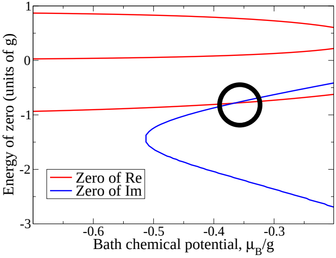


Zeros of $A(\omega)$, $B(\omega)$ and stability

- Near critical μ_B:

Consider ω near simultaneous zeros:

$$D^{R-1}(\omega) = (\omega - \xi) + i\alpha(\omega - \mu_{\text{eff}})$$



- Can write this as:

$$D^{R-1}(\omega) = (1+i\alpha)\left(\omega - \frac{\xi + i\alpha\mu_{\text{eff}}}{1+i\alpha}\right) = (1+i\alpha)(\omega - \omega^*)$$

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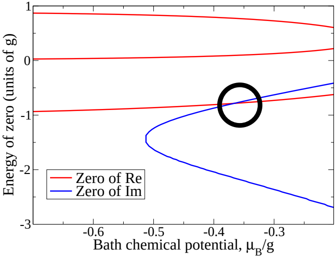

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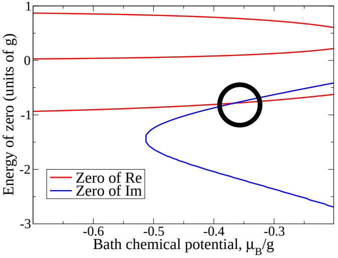

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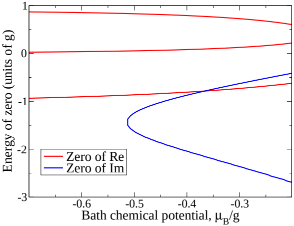

Weak pumping $B(\omega)$ always positive — no zero.

Subcritical Zero at μ_{eff} exists, but $\mu_{\text{eff}} < \xi$ so $\Im(\omega^*) < 0$.

Critical pumping Sufficient range of positive $B(\omega)$ s.t. $\mu_{\text{eff}} = \xi$ — real, non-decaying pole of Green's function. Diverging Luminescence (at $A(\omega^*) = B(\omega^*) = 0$).

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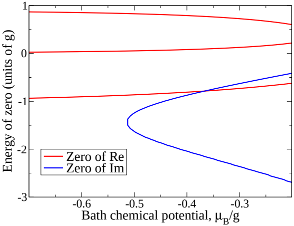

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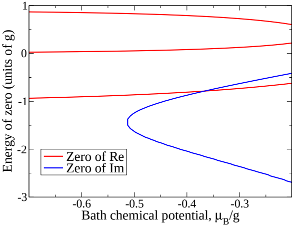

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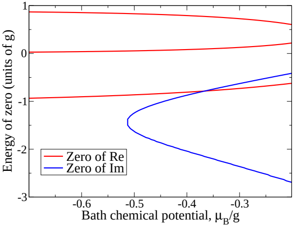

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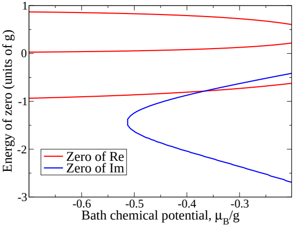

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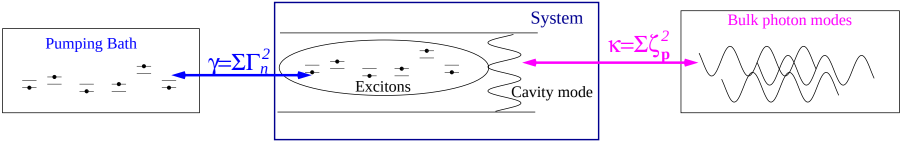
Overview

Maxwell Bloch equations



$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i \left(b_i^\dagger b_i - a_i^\dagger a_i \right) + \sum_{i,k} g \left(\psi_k^\dagger a_i^\dagger b_i + \text{H.c.} \right).$$

Semiclassical equations for: field ψ, polarisation $P = n\langle -i a^\dagger b\rangle$, inversion $N = n\langle b^\dagger b - a^\dagger a\rangle$

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
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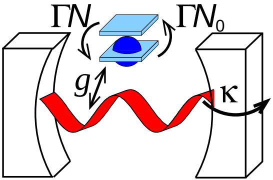

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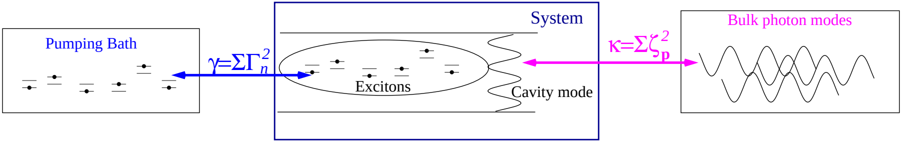

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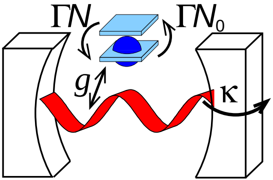
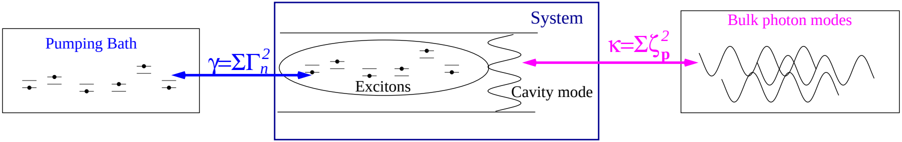


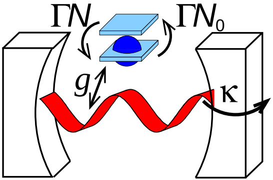
Maxwell Bloch equations



$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i \left(b_i^\dagger b_i - a_i^\dagger a_i \right) + \sum_{i,k} g \left(\psi_k^\dagger a_i^\dagger b_i + \text{H.c.} \right).$$

Semiclassical equations for: field ψ, polarisation $P = n\langle -ia^\dagger b\rangle$, inversion $N = n\langle b^\dagger b - a^\dagger a\rangle$

$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
$$\partial_t P = -2i\epsilon P - \Gamma P + g\psi N$$
$$\partial_t N = \Gamma(N_0 - N) - 2g(\psi^* P + P^* \psi),$$



Maxwell-Bloch inverse Green's function

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\end{aligned}$$

- Green's function: response to driving field:

$$\psi = iD^R(\omega) F e^{-i\omega t}, \qquad P = \chi(\omega)\psi,$$

- Substituting gives:

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- Substituting gives:

$$iD^R(\omega)\left[-i\omega + i\omega_k + \kappa - g\chi(\omega)\right] = 1$$

- Hence: $\left[D^R(\omega)\right]^{-1} = A(\omega) + iB(\omega) = \omega - \omega_k + i\kappa + \frac{g^2 N_0}{\omega - 2\epsilon + i\Gamma}$

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Maxwell-Bloch: Zeros of inverse Green's function

$$A(\omega) + iB(\omega) = \omega - \omega_k + i\kappa + \frac{g^2 N_0 \left[(\omega - 2\epsilon) - i\Gamma\right]}{(\omega - 2\epsilon)^2 + \Gamma^2}$$

For zero $B(\omega)$ need $N_0 > 0$. For zero $A(\omega)$ $N_0 > 0$ kills splitting.

For illustration, $\omega_k = 2\epsilon$:

$$A(\omega) = (\omega - 2\epsilon)\left[1 + \frac{g^2 N_0}{(\omega - 2\epsilon)^2 + \Gamma^2}\right]$$

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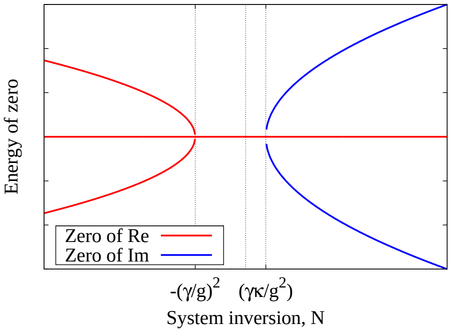

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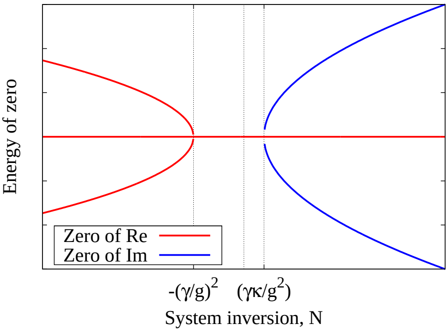
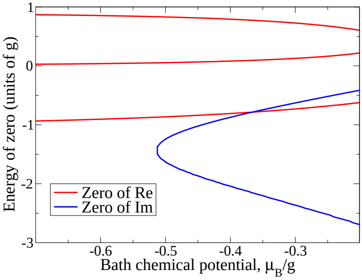
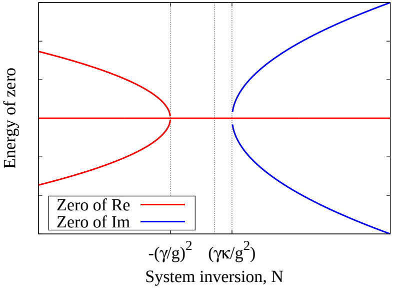


Comparing non-equilibrium condensate & LASER





$$\Downarrow$$

$$B(\omega) = \kappa - \frac{g^2 N_0 \Gamma}{(\omega - 2\epsilon)^2 + \Gamma^2}$$

$$B(\omega) = \kappa + g^2 n \gamma^2 \int \frac{d\nu}{2\pi} \frac{\tanh\left[\frac{\beta}{2}(\nu + \omega - \mu_B)\right] - \tanh\left[\frac{\beta}{2}(\nu + \mu_B)\right]}{\left[(\nu + \omega - \epsilon_i)^2 + \gamma^2\right]\left[(\nu + \epsilon_i)^2 + \gamma^2\right]},$$

Relating Keldysh & Maxwell-Bloch

- Heisenberg-Langevin correspond to:

$$\tanh\left[\frac{\beta}{2}(\nu+\omega-\mu_B)\right]-\tanh\left[\frac{\beta}{2}(\nu+\mu_B)\right]\to -2N_{\text{bath}}$$

- $B(\omega)$ reduces to:

$$B(\omega)=\kappa+g^2 n\gamma\int\frac{d\nu}{2\pi}\frac{-2N_{\text{bath}}\gamma}{\left[(\nu+\omega-\epsilon_i)^2+\gamma^2\right]\left[(\nu+\epsilon_i)^2+\gamma^2\right]}$$

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- Non-equilibrium diagram approach illustates continuum:

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- Non-equilibrium diagram approach illustates continuum:
  - LASER: transition when gain exceeds loss; requires inversion.
  - Non-equilibrium polariton condensate — gain exceeds loss at polariton modes.
  - Equilibrium condensate — transition when μ reaches bottom of band

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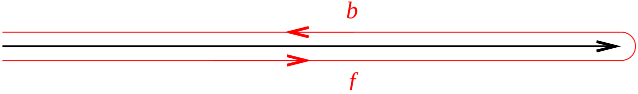
Summary

- Introduce non-equilibrium diagram approach



- Applied this to modelling system with particle flux .
- Discussed physical content of Green's functions
- Show connection between equilibrium condensate → non-equilibrium condensate → LASER.

Clermont 4 PhD position, October 2010 — Non-equilibrium phase transitions in 2D system. `jmjk2@cam.ac.uk`

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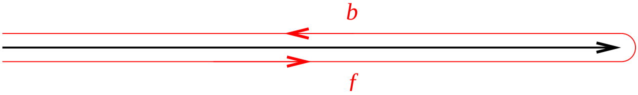

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Overview

4. Comparison to other non-equilibrium approaches
   - Boltzmann equation
   - Heisenberg-Langevin equations
   - Density matrix equations

Quantum Kinetic (Boltzmann) equation

- Decsribes rate of scattering into & out of states.
- Can relate to Keldysh GF: $\left[D^R\right]^{-1} D^K + \left[D^{-1}\right]^K D^A = 0$.
  Use parmaterisation: $D^K(\omega, k) = D^R(\omega, k)F(\omega) - F(\omega)D^A(\omega, k)$
- Rewrite equation as: $\left[D^R\right]^{-1} D^K \left[D^A\right]^{-1} = -\left[D^{-1}\right]^K$
  Separate $[D_0^R]^{-1} = i\partial_t - \nabla^2/2m$

[See, e.g. Lifshitz and Pitaevskii *Phyiscal Kinetics* Butterworth-Heinemann].

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$$\frac{\partial F}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla F = I_{\text{in}} - I_{\text{out}}(F)$$

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Heisenberg-Langevin equations

- Consider photon-bath coupling: $H_{\text{int}} = \sum_{k,P} \zeta_{k,p} \left(\psi^\dagger_k \Psi_p + \Psi^\dagger_p \psi_k \right),$
- Heisenberg equation: $i\partial_t \Psi_p = \zeta_{k,p}\psi_k + \omega^\zeta_k \Psi_p$
- Solution: $\Psi_p(t) = \Psi_p(0)e^{-i\omega^\zeta_p t} - i\zeta_{p,k}\int^t dt' \psi_k(t') e^{-i\omega^\zeta_p (t-t')}$
- Substitute in: $i\partial_t \psi_k = \sum_p \zeta_{p,k}\Psi_p + \ldots$

  In Markov approximation, $\sum_{p,k} \zeta^2_{p,k} e^{-i\omega^\zeta_p (t-t')} = 2\kappa\delta(t-t')$

  (*cf* bath retarded Green's function).
- Gives result: $i\partial_t \psi_k = \eta_\Psi - i\kappa\psi_k + \ldots$

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[See, e.g. Gardiner and Zoller *Quantum Noise*, Springer 2004].

Density matrix equation of motion

- Markov approximation allows:

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- Different focus

  Non-equilibrium diagrams Non-equal time one particle correlations.
  Many particles require extra diagrams.

  Density matrix Equal time multiparticle correlations.
  Multi-time correlations requires "Quantum regression theorem."

- Solve numerically by mapping ρ to quasiprobability (positive function)
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