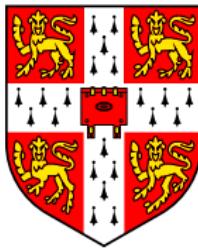


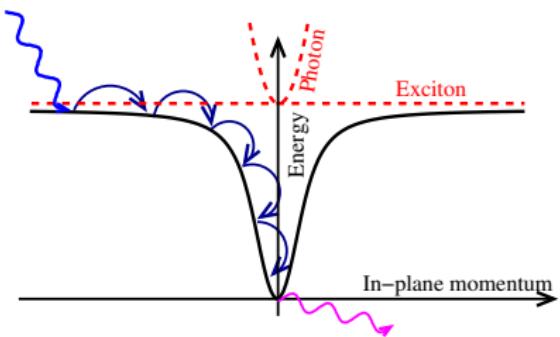
Comparing non-equilibrium polariton condensation and lasing

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International School of Nanophotonics and Photovoltaics
Maratea, September 2009

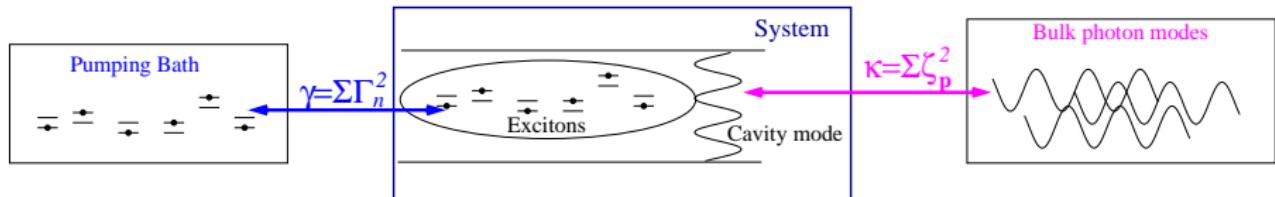
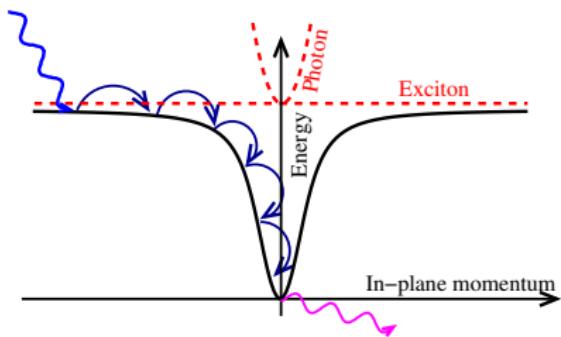


Aims



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- Non-equilibrium diagram (Keldysh) approach.
- Modelling system with particle flux.
- Show connection between:
 - ▶ Equilibrium BEC: chemical potential $\mu \rightarrow 0$.
 - ▶ Simple LASER: round-trip gain exceeds loss.



Overview

1 Non-equilibrium Green's functions

- Definitions of Green's function
- Diagrammatic approach
- Example: decaying photons

2 Polariton condensation and lasing

- Self energy for the Dicke model
- Analytic properties of Green's functions
- Normal state instability

3 Summary

Need for multiple Green's functions

- Consider definitions:

$$iD^<(t, r) = \langle \psi^\dagger(0, 0)\psi(t, r) \rangle, \quad iD^>(t, r) = \langle \psi(t, r)\psi^\dagger(0, 0) \rangle,$$

- Fourier transform:

$$iD^<(\omega, k) = n_B(\omega)p(\omega, k), \quad iD^>(\omega, k) = [n_B(\omega) + 1]p(\omega, k).$$

- In equilibrium, $n_B(\omega) = \frac{1}{e^{\beta\omega} - 1}$

- Out of equilibrium, need (at least) two Green's functions.

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Retarded and Keldysh Green's functions

- In fact, consider:

$$D^R(t, r) = -i\theta(t) \left\langle [\psi(t, r), \psi^\dagger(0, 0)] \right\rangle, \quad D^A(t, r) = [D^R(-t, -r)]^*$$
$$D^K(t, r) = -i \left\langle [\psi(t, r), \psi^\dagger(0, 0)]_+ \right\rangle.$$

• By definition

$$\psi(t, r) = \sum_k \psi_k e^{-i\omega_k t - ik\cdot r}.$$

• Hence

$$D^R(t, r) = -i\theta(t) \sum_k e^{-i\omega_k t - ik\cdot r}, \quad D^R(\omega, k) = \frac{1}{\omega - \omega_k + i0}$$

$$D^R(\omega, k) = -(2\pi)i(\omega - \omega_k)(2n_R(\omega_k) + 1)$$

• Density of states

$$D^R(\omega, k) - D^A(\omega, k) = -(2\pi)i(\omega - \omega_k),$$

• Can rewrite Keldysh Green's function

$$D^K(\omega, k) = [D^R(\omega, k) - D^A(\omega, k)](2n_R(\omega_k) + 1)$$

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- For free fields:

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- ▶ Definition of D^A :

$$D^R(\omega, k) - D^A(\omega, k) = -(2\pi i)\delta(\omega - \omega_k),$$

- ▶ Can rewrite Keldysh Green's function:

$$D^A(\omega, k) = [D^R(\omega, k) - i\delta(\omega, k)(2n_B(\omega) + 1)]$$

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Constructing diagrammatic expansion

- Need Heisenberg picture $\psi(t)$:

$$\psi(t) = e^{iHt} \psi e^{-iHt} = U^{-1}(t) \tilde{\psi}(t) U(t)$$

Interaction picture: $H = H_0 + H_{\text{int}}$ $\tilde{\psi}(t) = e^{iH_0 t} \psi e^{-iH_0 t}$,

- Gives equation for $U(t)$:

$$D_t U = -i e^{iH_0 t} H_{\text{int}} e^{-iH_0 t} U, \quad U(t) = T \exp \left[-i \int H_{\text{int}}(t') dt' \right].$$

- Aim: Green's function as time-ordered expectation

$$\langle T[A(t)B(t')] \rangle = \begin{cases} (A(t)B(t)) & t > t' \\ (B(t')A(t)) & t < t' \end{cases}$$

- In time ordered products: $D = \langle T [A(t)B(t')] \rangle = \langle T [A(t)B(t')U] \rangle$

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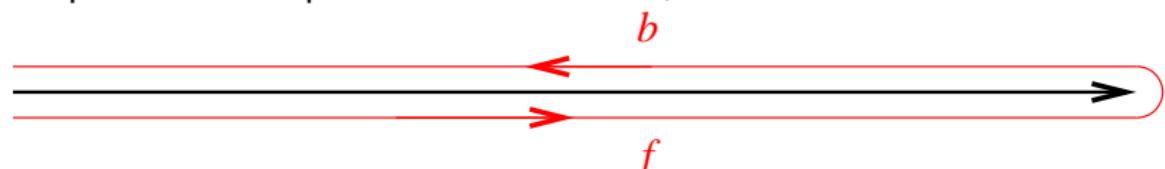
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Keldysh contour

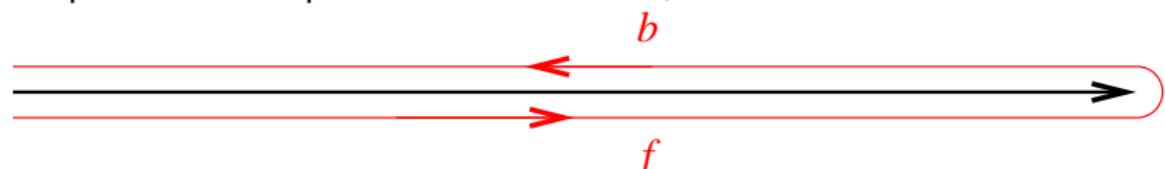
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Ordering T_c : $(t, b) > (t', f)$ for all t, t' .

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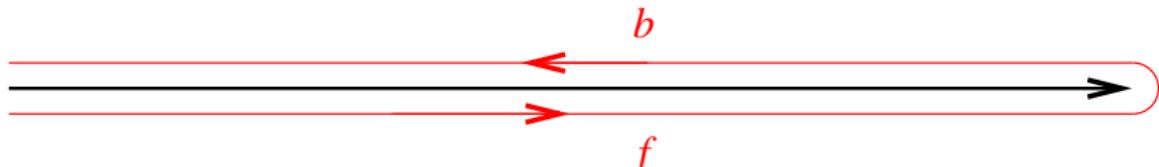
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- Hence: $D^>(t) = -i\langle \psi(t)\psi^\dagger(0) \rangle = -i \left\langle T_c (\psi(t, b)\psi^\dagger(0, f)) \right\rangle$.

Keldysh contour and Green's functions



- Define $\psi_{\pm}(t) = \frac{1}{\sqrt{2}} (\psi(t, f) \pm \psi(t, b))$,

• Gives retarded and advanced functions

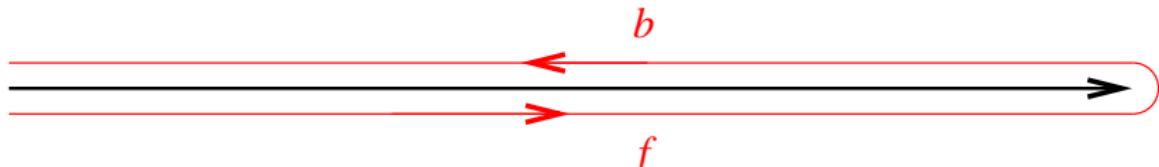
$$D = \begin{pmatrix} D^R & D^A \\ D^A & 0 \end{pmatrix} = -i \left(T_C \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \begin{pmatrix} \psi_+(0) & \psi_-(0) \end{pmatrix} \right)$$

• For example:

$$D^R = -\frac{i}{2} \left(T_C [(\psi(0, f) + \psi(0, b))(\psi(0, f) - \psi(0, b))] \right)$$

$$= -\frac{i}{2} \left(\psi(0, f)\psi(0, b) - \psi(0, b)\psi(0, f) + (\psi(0)\psi'(0) - \psi'(0)\psi(0)) \delta(t) \right) \quad t > 0$$
$$= \psi(0)\psi'(0) - \psi'(0)\psi(0) \quad t < 0$$

Keldysh contour and Green's functions



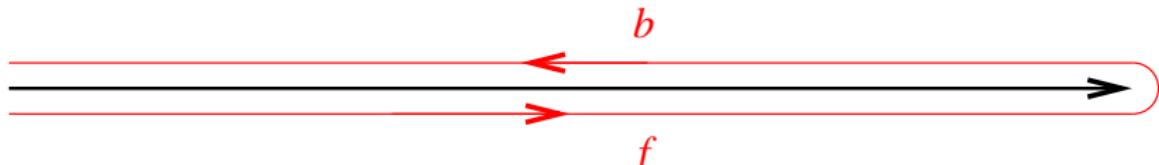
- Define $\psi_{\pm}(t) = \frac{1}{\sqrt{2}} (\psi(t, f) \pm \psi(t, b))$,
- Gives matrix of Green's functions:

$$D = \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = -i \left\langle T_c \begin{pmatrix} \psi_+(t) \\ \psi_-(t) \end{pmatrix} \begin{pmatrix} \psi_+^\dagger(0) & \psi_-^\dagger(0) \end{pmatrix} \right\rangle$$

For example

$$\begin{aligned} D^R &= -\frac{i}{2} \left\langle T_c \left[(\psi(t, f) + \psi(t, b)) (\psi^\dagger(0, f) - \psi^\dagger(0, b)) \right] \right\rangle \\ &= -\frac{i}{2} \left(\psi(t) \psi^\dagger(0) - \psi(t) \psi^\dagger(0) + \left(\psi(t) \psi^\dagger(0) - \psi^\dagger(0) \psi(t) - \psi(t) \psi^\dagger(0) + \psi^\dagger(0) \psi(t) \right) \right) \quad t > 0 \end{aligned}$$

Keldysh contour and Green's functions



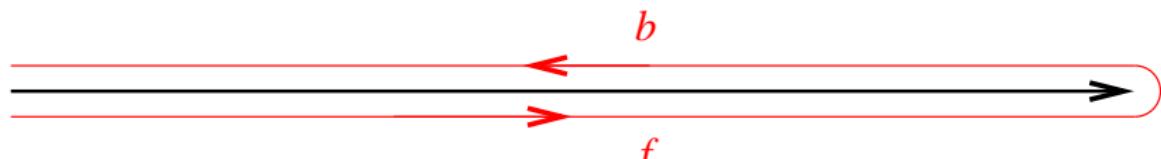
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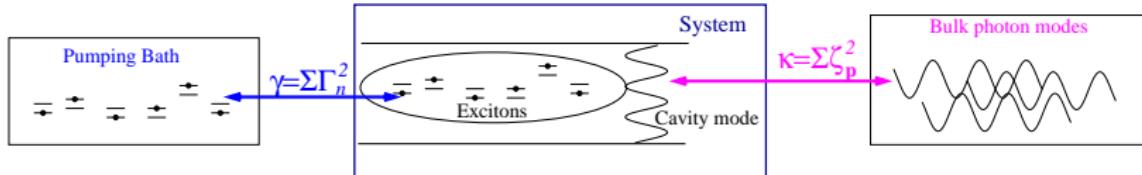
Non-equilibrium diagrammatic theory: Summary



- $\begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = -i \left\langle T_c \left[\begin{pmatrix} \tilde{\psi}_+(t) \\ \tilde{\psi}_-(t) \end{pmatrix} \left(\tilde{\psi}_+^\dagger(0) \quad \tilde{\psi}_-^\dagger(0) \right) U \right] \right\rangle$
- Propagator:

$$\begin{aligned} U &= \exp \left(-i \int_C \tilde{H}_{\text{int}}(t) dt \right) \\ &= \exp \left(-i \int_{-\infty}^{\infty} [\tilde{H}_{\text{int}}(t, f) - \tilde{H}_{\text{int}}(t, b)] dt \right). \end{aligned}$$

Illustration: coupling to a decay bath



• Coupling: $H_{\text{int}} = \sum_{kp} g_{kp} (\hat{a}_k^\dagger \Psi_p + \Psi_p^\dagger \hat{a}_k)$

• To find propagator need

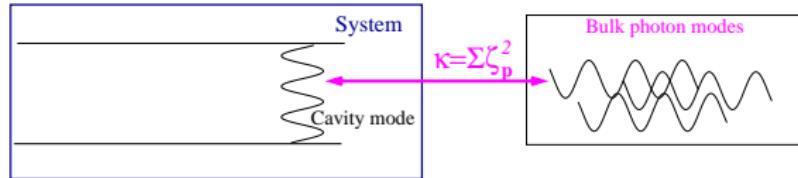
$$\int_C P_{\text{int}}(t) = \sum_{kp} g_{kp} \int_{-\infty}^t [\hat{a}_{k+}^\dagger \Psi_{p-} + \hat{a}_{k-}^\dagger \Psi_{p+} + H.c.]$$

• Want to resum perturbation series:

• To resum, use

$$D = D_0 + D_0 \sum D \quad D^{-1} = D_0^{-1} - \Sigma$$

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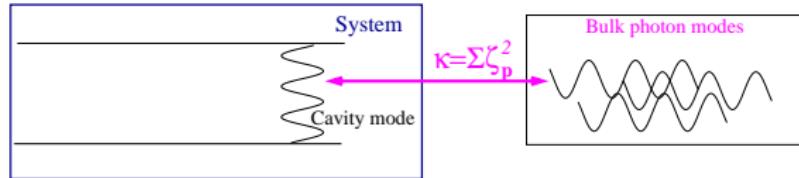
$$\int_C P_{\text{out}}(t) = \sum_{kp} \zeta_{kp} \int_{-\infty}^t [\hat{a}_{k+}^\dagger \psi_{p-} + \hat{a}_{k-}^\dagger \psi_{p+} + H.c.]$$

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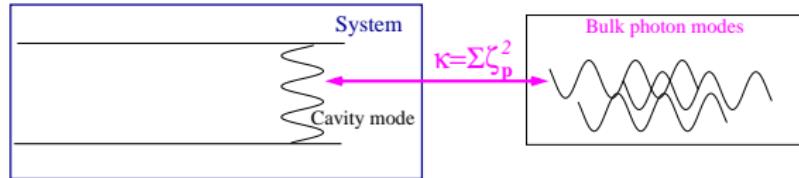
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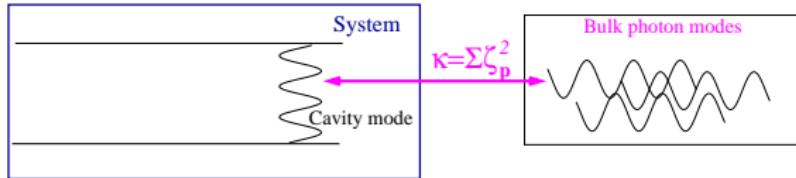
$$\int_C \tilde{H}_{\text{int}}(t) = \sum_{k,P} \zeta_{k,p} \int_{-\infty}^{\infty} [\psi_{k,+}^\dagger \psi_{p,-} + \psi_{k,-}^\dagger \psi_{p,+} + \text{H.c.}] .$$

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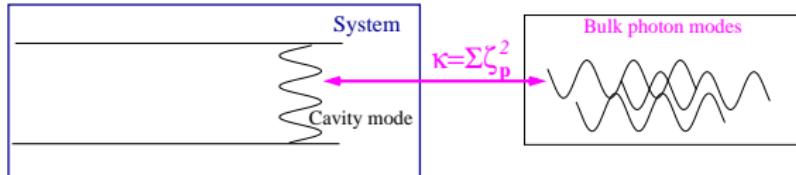
$$\sim\!\!\sim = \sim\!\!\sim + \sim\!\!\sim \zeta \dots \zeta \sim\!\!\sim + \sim\!\!\sim \zeta \dots \zeta \sim\!\!\sim \zeta \dots \zeta \sim\!\!\sim +$$

To resum, use

$$D = D_0 + D_0 \sum D$$

$$D^{-1} = D_0^{-1} - \Sigma$$

Illustration: coupling to a decay bath



- Coupling: $H_{\text{int}} = \sum_{k,P} \zeta_{k,p} (\psi_k^\dagger \psi_p + \psi_p^\dagger \psi_k)$
- To find propagator need:

$$\int_C \tilde{H}_{\text{int}}(t) = \sum_{k,P} \zeta_{k,p} \int_{-\infty}^{\infty} [\psi_{k,+}^\dagger \psi_{p,-} + \psi_{k,-}^\dagger \psi_{p,+} + \text{H.c.}] .$$

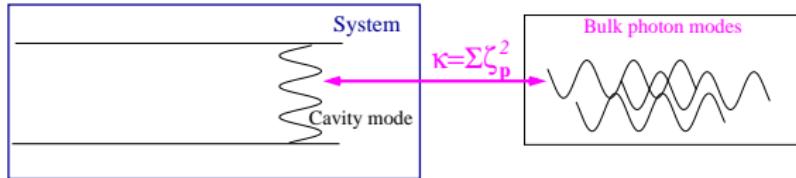
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Dyson equation, self energy

- Inverse Green's function

$$D^{-1} = D_0^{-1} - \Sigma = \begin{pmatrix} 0 & \omega - \omega_k - i0 \\ \omega - \omega_k + i0 & i0 \dots \end{pmatrix} - \begin{pmatrix} 0 & \Sigma^R \\ \Sigma^A & \Sigma^K \end{pmatrix}.$$

To solve, use:

$$\begin{pmatrix} 0 & [D^A]^\dagger \\ [D^A]^{-1} & [D^{-1}]^K \end{pmatrix} \begin{pmatrix} D^K & D^R \\ D^A & 0 \end{pmatrix} = 0$$
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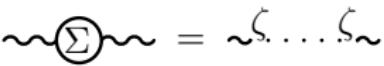
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Self energy and Markov approximation

$$\int_C \tilde{H}_{\text{int}}(t) = \sum_{k,p} \zeta_{k,p} \int_{-\infty}^{\infty} [\psi_{k,+}^\dagger \Psi_{p,-} + \psi_{k,-}^\dagger \Psi_{p,+} + \text{H.c.}] .$$

e.g. Retarded self energy  = $\sim \zeta \dots \zeta \sim$

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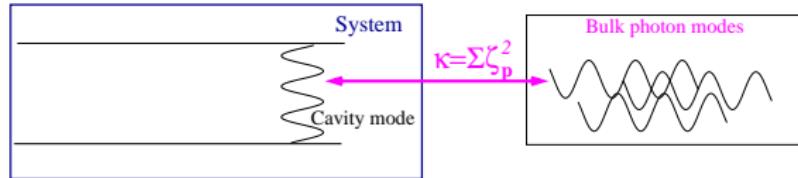
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Decaying Green's functions



- Inverse Green's functions:

$$\begin{pmatrix} 0 & [D^A]^{-1} \\ [D^R]^{-1} & [D^{-1}]^K \end{pmatrix} = \begin{pmatrix} 0 & \omega - \omega_k - i\kappa \\ \omega - \omega_k + i\kappa & 2i\kappa(2n_\Psi + 1) \end{pmatrix}$$

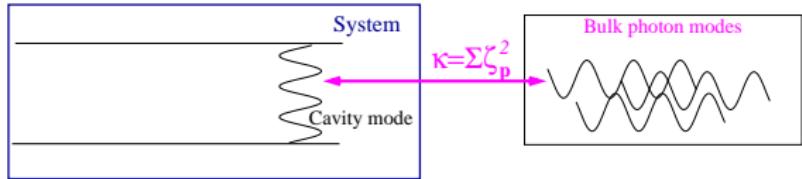
- Green's functions

$$D^S(\omega, k) = \frac{1}{\omega - \omega_k + i\kappa}$$

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- Lorentzian broadening, population set by bath

Decaying Green's functions



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Overview

1 Non-equilibrium Green's functions

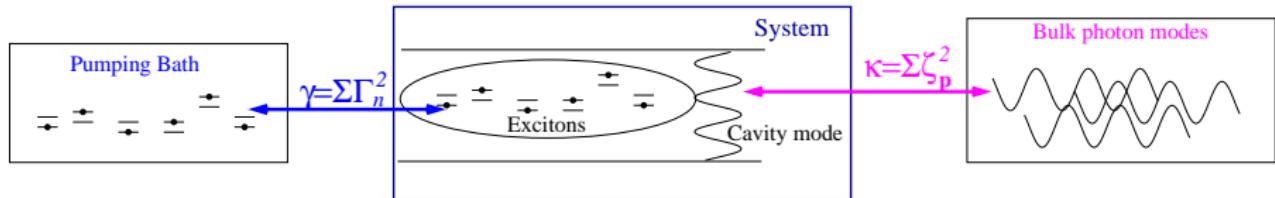
- Definitions of Green's function
- Diagrammatic approach
- Example: decaying photons

2 Polariton condensation and lasing

- Self energy for the Dicke model
- Analytic properties of Green's functions
- Normal state instability

3 Summary

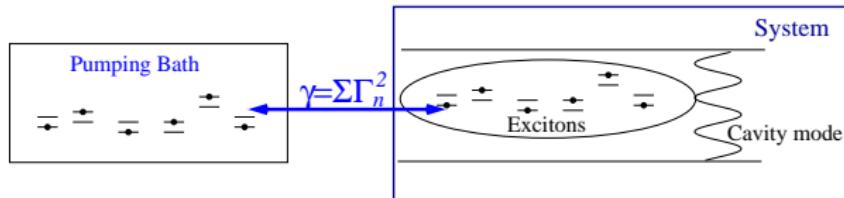
Two-level system model of polaritons



• Two-level system Hamiltonian

$$H = \sum_k \omega_k b_k^\dagger b_k + \sum_i \epsilon_i (b_i^\dagger b_i - a_i^\dagger a_i) + \sum_{jk} g_j (a_j^\dagger a_j^\dagger b_k + H.c.)$$

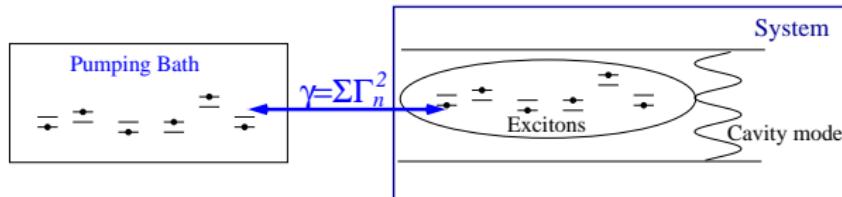
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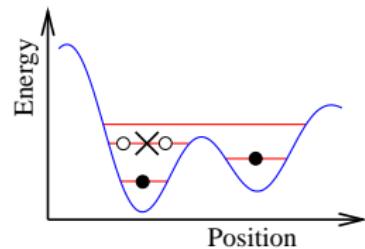
• Two-level system Hamiltonian

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Two-level system model of polaritons



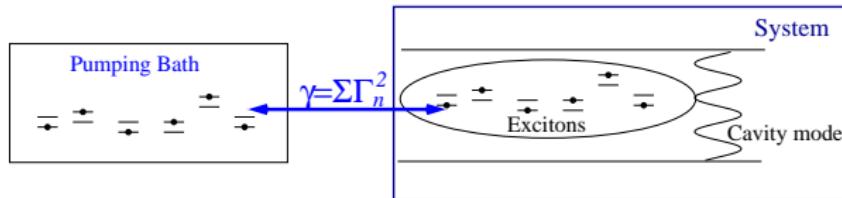
- - ▶ Disorder-localised excitons
 - ▶ Treat sites as two-level system (exciton/no-exciton)
 - ▶ Propagating (2D) photons
 - ▶ Exciton–photon coupling g .



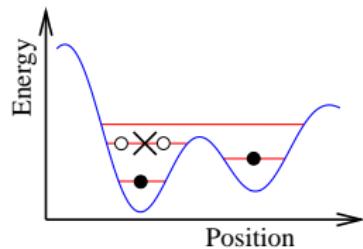
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$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i (b_i^\dagger b_i - a_i^\dagger a_i) + \sum_{i,k} g (\psi_k^\dagger a_i^\dagger b_i + \text{H.c.}) .$$

Polariton self energy

- Include repeated exciton \leftrightarrow photon interconversion use

$$\text{wavy wavy} = \text{wavy wavy} + \text{wavy wavy} \circledcirc \Sigma.$$

- Split free/**interaction** parts:

$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i (b_i^\dagger b_i - a_i^\dagger a_i) + \sum_{i,k} g (\psi_k^\dagger a_i^\dagger b_i + \text{H.c.})$$

- Add $(\ell, b) \rightarrow (+, -)$ labels to operators

$$\begin{aligned} \int_C dt \tilde{H}_{\text{int}} = & \sum_{i,k} dt \frac{\pi}{\sqrt{2}} [\tilde{a}_+(t) (\tilde{b}_{++}(t) \tilde{a}_{++}(t) + \tilde{b}_{+-}(t) \tilde{a}_{+-}(t)) \\ & + \tilde{a}_-(t) (\tilde{b}_{++}(t) \tilde{a}_{++}(t) + \tilde{b}_{+-}(t) \tilde{a}_{+-}(t)) + \text{H.c.}] \end{aligned}$$

Summing terms all have odd number "x" terms.

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Surviving terms all have odd number “-” terms.

Polariton retarded self energy

- Retarded self energy $\Sigma_{\psi^\dagger \psi}^{-+} =$

- Using fermion Green's functions G :

$$\Sigma_{\psi^\dagger \psi}^{-+} = -i \left(\frac{g}{\sqrt{2}} \right)^2 \int \frac{du}{2\pi} \sum [G_{\psi \psi}^A(u) G_{\psi \psi}^K(u+\omega) + G_{\psi \psi}^K(u) G_{\psi \psi}^A(u+\omega)]$$

- Fermions coupled to pumping bath:

$$G_{\text{pump}}^R = \frac{1}{u + \alpha + i\gamma} \quad G_{\text{pump}}^K = \frac{2i\gamma \tanh \left[\frac{\beta}{2} (u + \mu_B) \right]}{(u + \alpha)^2 + \gamma^2}$$

- β, μ_B describe pumping bath.

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$$G_{\text{pump}}^R = \frac{1}{\nu + \alpha + i\gamma}$$

$$G_{\text{pump}}^K = \frac{2i\gamma \tanh \left[\frac{\beta}{2} (\nu + \mu_B) \right]}{(\nu + \alpha)^2 + \gamma^2}$$

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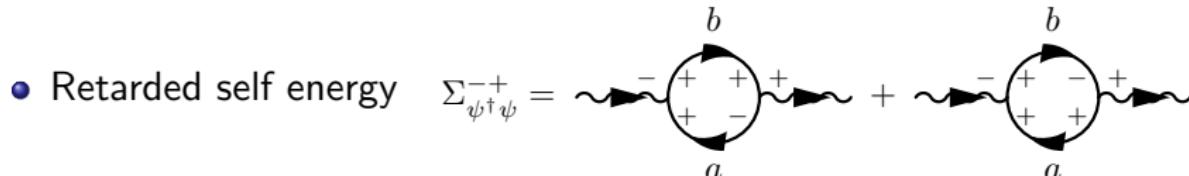
Fermions coupled to pumping bath:

$$G_{a^\dagger a}^R = \frac{1}{\nu - \epsilon_a + i\gamma}$$

$$G_{b^\dagger b}^K = \frac{2i\gamma \tanh \frac{\beta}{2}(\nu - \mu_B)}{(\nu - \epsilon_b)^2 + \gamma^2}$$

• But we describe pumping bath:

Polariton retarded self energy



- Using fermion Green's functions G :

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- Fermions coupled to pumping bath:

$$G_{b^\dagger b, a^\dagger a}^R = \frac{1}{\nu \mp \epsilon_i + i\gamma}, \quad G_{b^\dagger b, a^\dagger a}^K = -\frac{2i\gamma \tanh \left[\frac{\beta}{2} (\nu \pm \mu_B) \right]}{(\nu \mp \epsilon_i)^2 + \gamma^2},$$

- β, μ_B describe pumping bath.

Inverse polariton Green's function

$$\begin{aligned} \left[D^R(\omega, k) \right]^{-1} &= \omega - \omega_k + i\kappa \\ &+ g^2 n \gamma \int \frac{d\nu}{2\pi} \frac{\tanh \left[\frac{\beta}{2} (\nu + \omega - \mu_B) \right]}{\left[(\nu + \omega - \epsilon_i)^2 + \gamma^2 \right] (\nu + \epsilon_i - i\gamma)} \\ &+ g^2 n \gamma \int \frac{d\nu}{2\pi} \frac{\tanh \left[\frac{\beta}{2} (\nu + \mu_B) \right]}{(\nu + \omega - \epsilon_i + i\gamma) \left[(\nu + \epsilon_i)^2 + \gamma^2 \right]}, \end{aligned}$$

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Meaning of poles of Green's functions

- Write: $[D^R(\omega)]^{-1} = A(\omega) + iB(\omega)$ $[D^{-1}(\omega)]^K = iC(\omega)$,

$$\text{Find } D^S(\omega) = \frac{1}{2} [D^K(\omega) - (D^R(\omega) - D^A(\omega))] = \frac{iC(\omega) - 2B(\omega)}{2[A(\omega)^2 + B(\omega)^2]}$$

- Physical Luminescence:

$$iD^S(\omega) = \frac{1}{2} [D^K(\omega) - (D^R(\omega) - D^A(\omega))] = \frac{C(\omega) - 2B(\omega)}{2[A(\omega)^2 + B(\omega)^2]}$$

- Divide into DoS \times Occupation: $iD^S(\omega) = \rho_{\text{eff}}(\omega)n_{\text{eff}}(\omega)$

$$\rho_{\text{eff}}(\omega) = - (D^R(\omega) - D^A(\omega)) = \frac{2B(\omega)}{A(\omega)^2 + B(\omega)^2}$$

$$n_{\text{eff}}(\omega) = \frac{1}{2} \left[\frac{C(\omega)}{2B(\omega)} - 1 \right]$$

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Physical dimensions:

$$iD^S(\omega) = \frac{1}{2} [D^K(\omega) - (D^R(\omega) - D^A(\omega))] = \frac{C(\omega) - 2B(\omega)}{2[A(\omega)^2 + B(\omega)^2]}$$

Divide into DoFs \times Occupation: $iD^S(\omega) = \rho_{\text{eff}}(\omega) n_{\text{eff}}(\omega)$

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- Write: $[D^R(\omega)]^{-1} = A(\omega) + iB(\omega)$ $[D^{-1}(\omega)]^K = iC(\omega)$,
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• For decaying phonons:

$$A(\omega) = \omega - \omega_k \quad \text{Describes location of normal modes}$$

$$B(\omega) = \kappa \quad \text{Describes linewidths of these modes}$$

$$C(\omega) = 2\kappa(2n_B(\omega) + 1) \quad \text{Describes their occupation}$$

• With coupling to pumped excitons $B(\omega)$ can vary (and vanish) → effective chemical potential.

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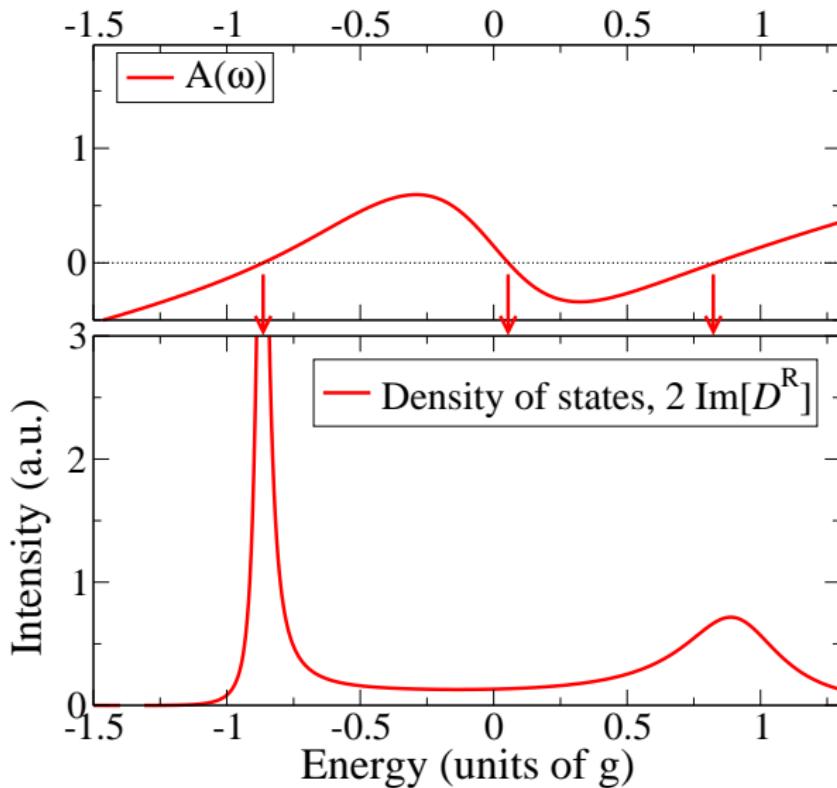
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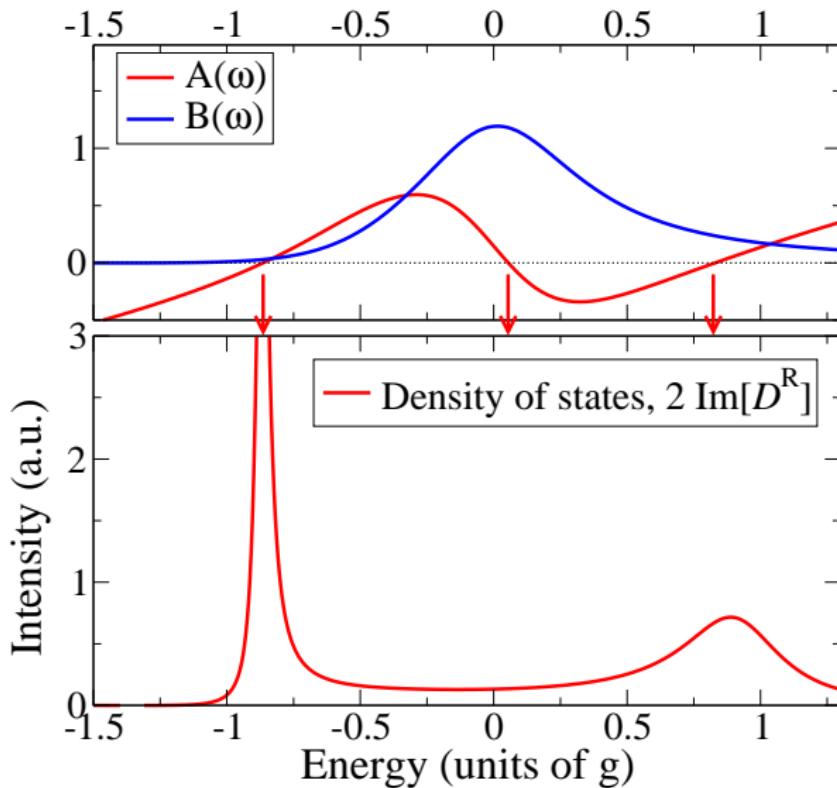
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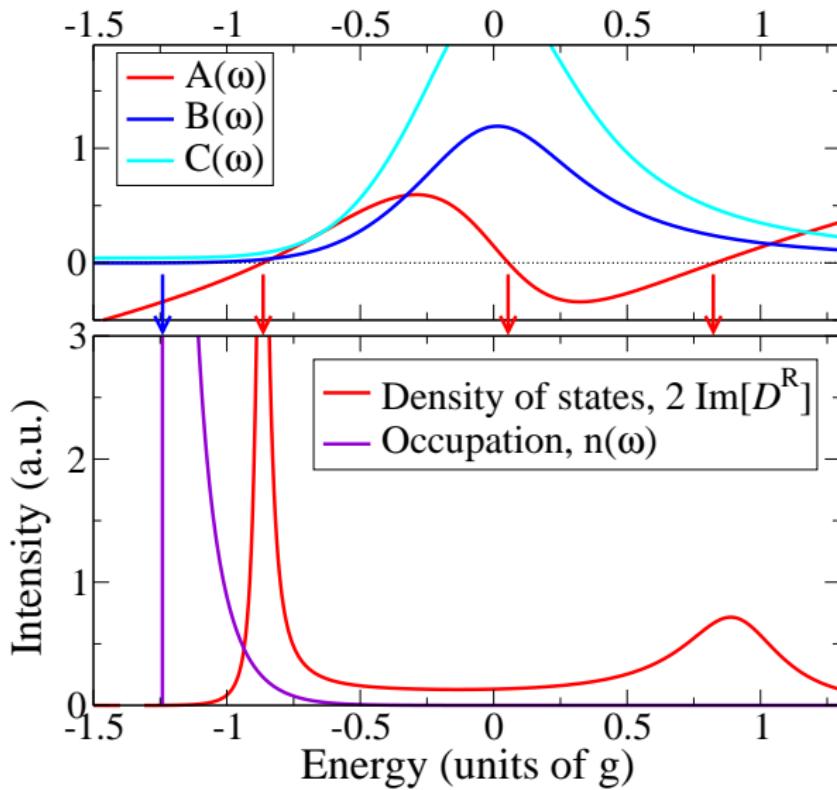
Inverse polariton Green's functions



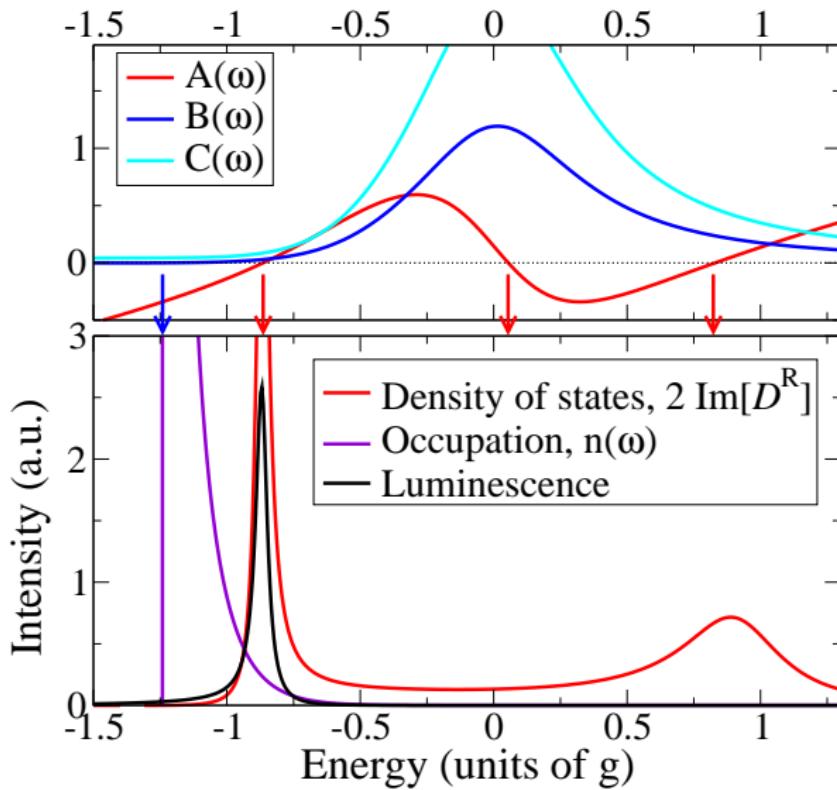
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Evolution with pumping

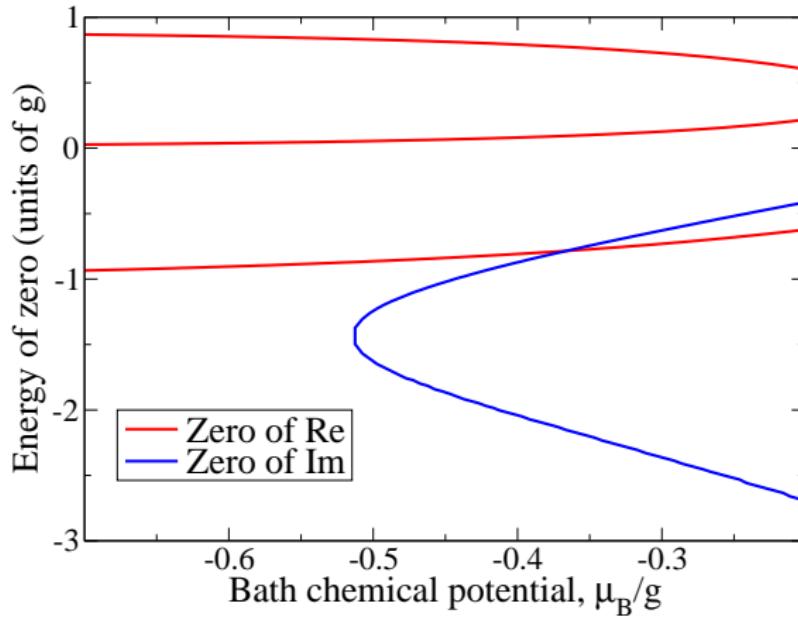
From before:

$$B(\omega) = \kappa + g^2 n \gamma^2 \int \frac{d\nu}{2\pi} \frac{\tanh \left[\frac{\beta}{2} (\nu + \omega - \mu_B) \right] - \tanh \left[\frac{\beta}{2} (\nu + \mu_B) \right]}{\left[(\nu + \omega - \epsilon_i)^2 + \gamma^2 \right] \left[(\nu + \epsilon_i)^2 + \gamma^2 \right]},$$

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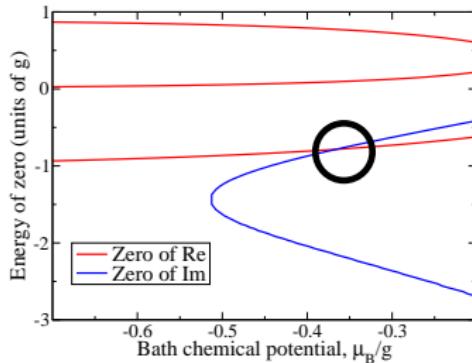


Zeros of $A(\omega)$, $B(\omega)$ and stability

- Near critical μ_B :

Consider ω near simultaneous zeros:

$$D^{R-1}(\omega) = (\omega - \xi) + i\alpha(\omega - \mu_{\text{eff}})$$



- Can write this as

$$D^{R-1}(\omega) = (1 + i\alpha) \left(\omega - \frac{\xi + i\alpha \mu_{\text{eff}}}{1 + i\alpha} \right) = (1 + i\alpha)(\omega - \omega^*)$$

- Thus, $D^R(i) \simeq e^{-i\omega^* t}$; need $\Im[\omega^*] < 0$ for stability, but

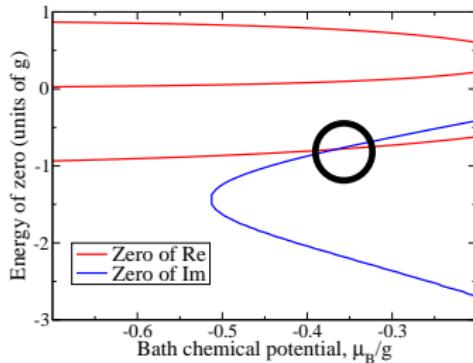
$$\omega^* = \frac{(\xi + \alpha^2/\alpha) + i\alpha(\mu_{\text{eff}} - \xi)}{1 + \alpha^2}$$

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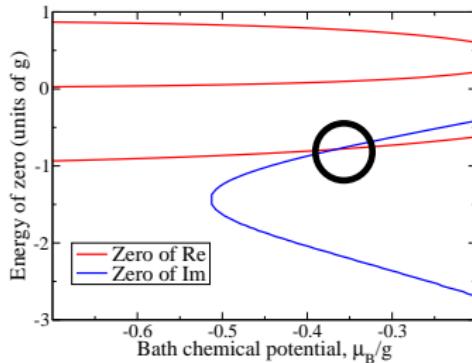
$$\Im[\omega^*] = \frac{(1 + \alpha^2)\tau_B + i\alpha(\omega^* - \xi)}{1 + i\alpha}$$

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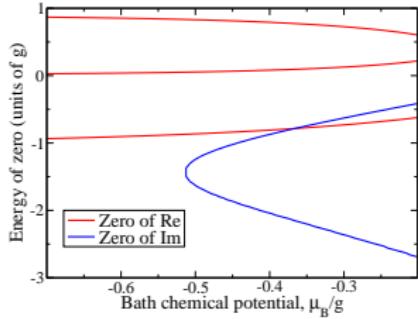
$$\omega^* = \frac{(\xi + \alpha^2 \mu_{\text{eff}}) + i\alpha(\mu_{\text{eff}} - \xi)}{1 + \alpha^2}.$$

Evolution with pumping

$$B(\omega) = \kappa + g^2 n \gamma^2 \int \frac{d\nu}{2\pi} \frac{\tanh\left[\frac{\beta}{2}(\nu + \omega - \mu_B)\right] - \tanh\left[\frac{\beta}{2}(\nu + \mu_B)\right]}{\left[(\nu + \omega - \epsilon_i)^2 + \gamma^2\right] \left[(\nu + \epsilon_i)^2 + \gamma^2\right]},$$

Weak pumping: $B(\omega)$ always positive — no zero.

Subcritical: Zero at μ_{eff} exists, but $\mu_{\text{eff}} < 0$ so $S(\omega) < 0$.



Critical pumping: Sufficient range of positive $B(\omega)$ ($\omega = \omega^*$) leads to a non-decaying pole of Green's function. Diverging Luminescence (at $A(\omega^*) = B(\omega^*) = 0$).

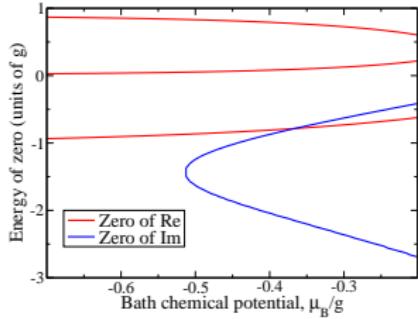
Supercritical: Non-condensed state unstable as $\mu_{\text{eff}} > 0$.

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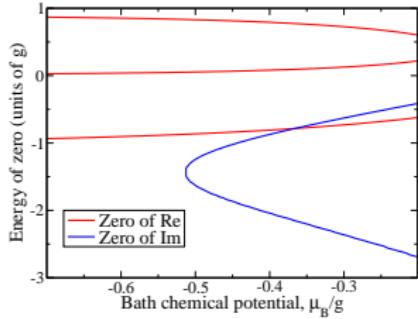
Supercritical: Non-condensed state unstable as $\mu_B \rightarrow 0$.

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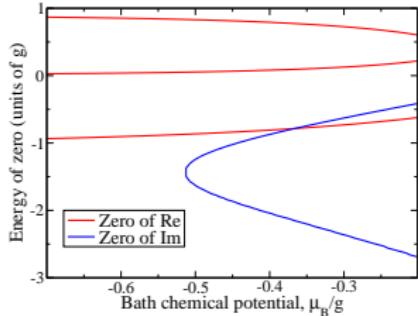
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Critical pumping Sufficient range of positive $B(\omega)$ s.t. $\mu_{\text{eff}} = \xi$ — real, non-decaying pole of Green's function. Diverging Luminescence (at $A(\omega^*) = B(\omega^*) = 0$).



Evolution with pumping

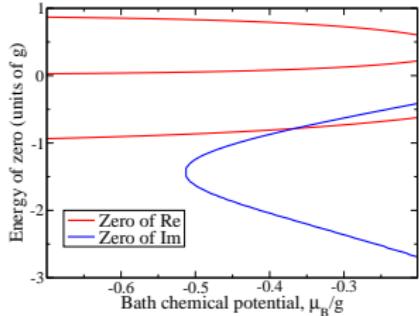
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Supercritical Non-condensed state unstable as $\mu_{\text{eff}} > \xi$.



Overview

1 Non-equilibrium Green's functions

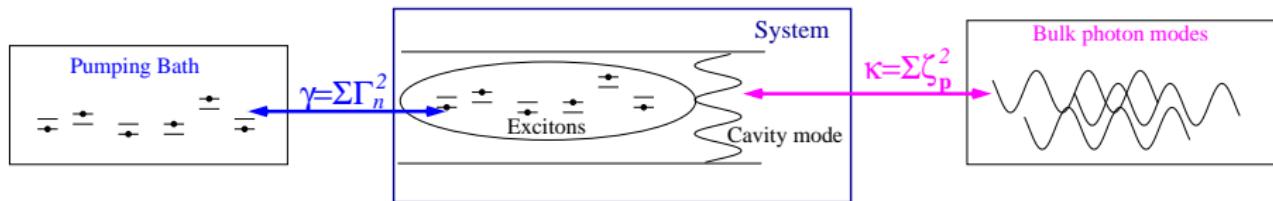
- Definitions of Green's function
- Diagrammatic approach
- Example: decaying photons

2 Polariton condensation and lasing

- Self energy for the Dicke model
- Analytic properties of Green's functions
- Normal state instability
 - Maxwell Bloch equations

3 Summary

Maxwell Bloch equations



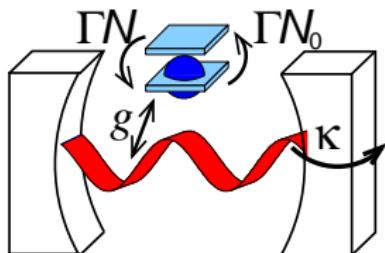
$$H = \sum_k \omega_k \psi_k^\dagger \psi_k + \sum_i \epsilon_i (b_i^\dagger b_i - a_i^\dagger a_i) + \sum_{i,k} g (\psi_k^\dagger a_i^\dagger b_i + \text{H.c.}) .$$

Semiclassical equations for field ψ , polarisation $P = n(-i\dot{a}^\dagger b)$, inversion $N = n(b^\dagger b - a^\dagger a)$

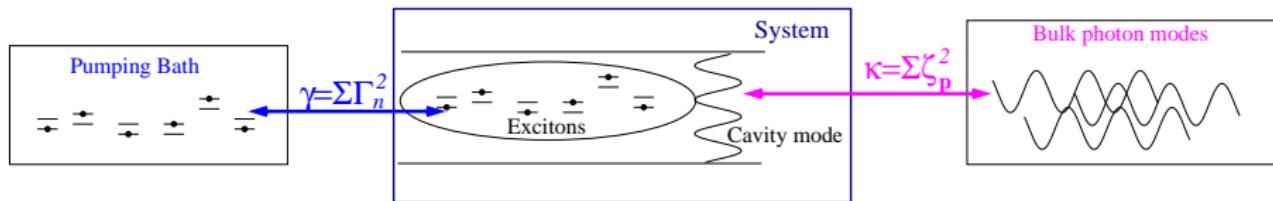
$$\partial_t \psi = -i\omega_k \psi - \kappa b + eP$$

$$\partial_t P = -2i\epsilon P - \Gamma P + g\psi N$$

$$\partial_t N = \Gamma(N_0 - N) - 2e(\partial_t P + \Gamma \psi)$$

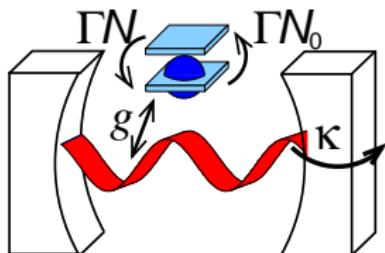


Maxwell Bloch equations

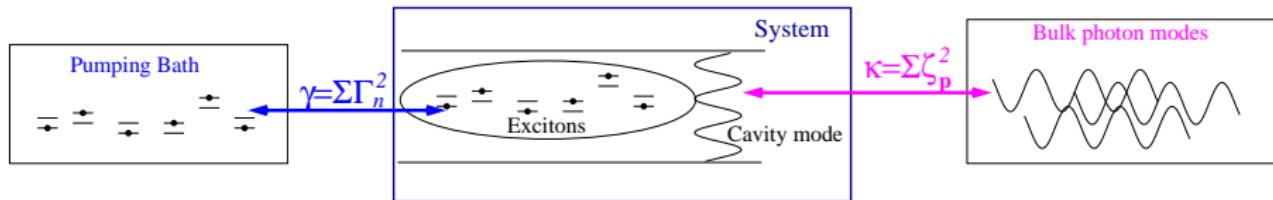


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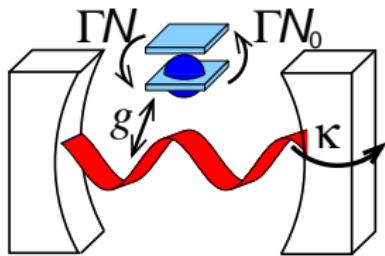
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Maxwell-Bloch inverse Green's function

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- Green's function: response to driving field:

$$\psi = \psi(\omega) e^{-i\omega t}, \quad P = P(\omega) e^{-i\omega t}$$

- Substituting gives:

Maxwell-Bloch inverse Green's function

$$\partial_t \psi = -i\omega_k \psi - \kappa \psi + gP + F e^{-i\omega t}$$

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Substituting gives

Maxwell-Bloch inverse Green's function

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- Substituting gives:

$$iD^R(\omega) [-i\omega + i\omega_k + \kappa - g\chi(\omega)] = 1$$

Hence $[D^R(\omega)]^{-1} = A(\omega) + iB(\omega) = \omega - \omega_k + i\kappa - g\chi(\omega)$

Maxwell-Bloch inverse Green's function

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- Substituting gives:

$$iD^R(\omega)[-i\omega + i\omega_k + \kappa - g\chi(\omega)] = 1$$

$$\chi(\omega)(-i\omega + 2i\epsilon + \Gamma) = gN_0.$$

Hence $[D^R(\omega)]^{-1} = A(\omega) + iB(\omega) = \omega - \omega_k + i\Gamma/2 + i\kappa/2$

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$$\chi(\omega)(-i\omega + 2i\epsilon + \Gamma) = gN_0.$$

- Hence: $[D^R(\omega)]^{-1} = A(\omega) + iB(\omega) = \omega - \omega_k + i\kappa + \frac{g^2 N_0}{\omega - 2\epsilon + i\Gamma}$

Maxwell-Bloch: Zeros of inverse Green's function

$$A(\omega) + iB(\omega) = \omega - \omega_k + i\kappa + \frac{g^2 N_0 [(\omega - 2\epsilon) - i\Gamma]}{(\omega - 2\epsilon)^2 + \Gamma^2}$$

For zero $B(\omega)$ need $N_0 > 0$. For
zero $A(\omega)$ $N_0 > 0$ kills splitting.

For illustration, $\omega_k = 2\epsilon$

$$A(\omega) = (\omega - 2\epsilon) \left[1 + \frac{g^2 N_0}{(\omega - 2\epsilon)^2 + \Gamma^2} \right]$$

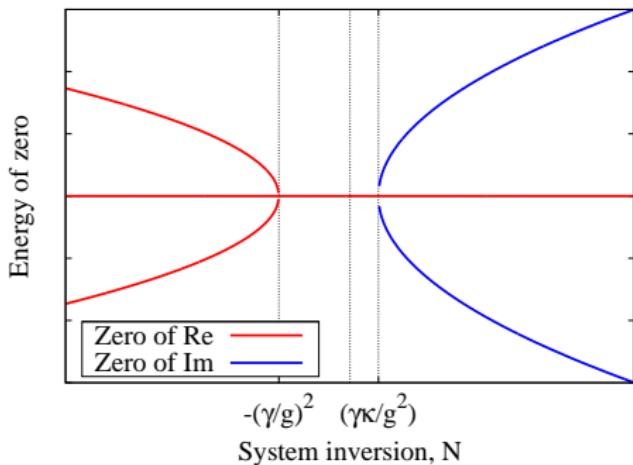
Maxwell-Bloch: Zeros of inverse Green's function

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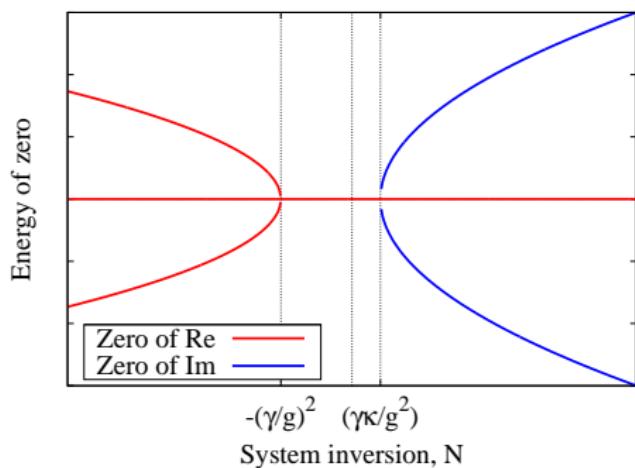
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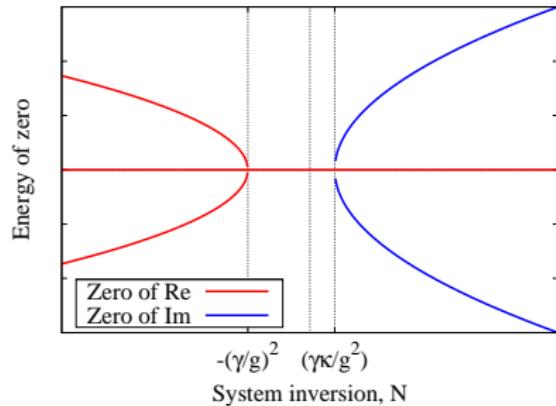
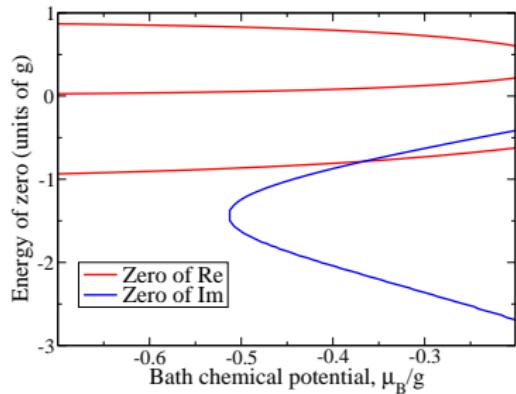
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Comparing non-equilibrium condensate & LASER



↓

$$B(\omega) = \kappa - \frac{g^2 N_0 \Gamma}{(\omega - 2\epsilon)^2 + \Gamma^2}$$

$$B(\omega) = \kappa + g^2 n \gamma^2 \int \frac{d\nu}{2\pi} \frac{\tanh \left[\frac{\beta}{2} (\nu + \omega - \mu_B) \right] - \tanh \left[\frac{\beta}{2} (\nu + \mu_B) \right]}{\left[(\nu + \omega - \epsilon_i)^2 + \gamma^2 \right] \left[(\nu + \epsilon_i)^2 + \gamma^2 \right]},$$

Relating Keldysh & Maxwell-Bloch

- Heisenberg-Langevin correspond to:

$$\tanh \left[\frac{\beta}{2} (\nu + \omega - \mu_B) \right] - \tanh \left[\frac{\beta}{2} (\nu + \mu_B) \right] \rightarrow -2N_{\text{bath}}$$

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 - Non-equilibrium polariton condensate — gain exceeds loss at polariton modes.
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Non-equilibrium polariton condensate → gain exceeds loss; no polariton production

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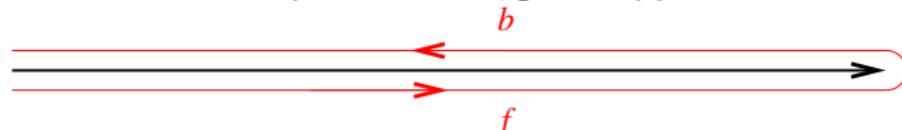
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Summary

- Introduce non-equilibrium diagram approach

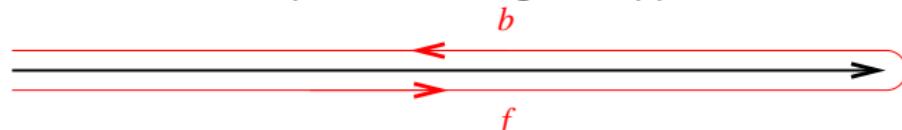


- Applied this to modelling system with particle flux .
- Discussed physical content of Green's functions
- Show connection between equilibrium condensate → non-equilibrium condensate → LASER.

Clermont 4 PhD position, October 2010 — Non-equilibrium phase transitions in 2D system. jmjk2@cam.ac.uk

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Overview

④ Comparison to other non-equilibrium approaches

- Boltzmann equation
- Heisenberg-Langevin equations
- Density matrix equations

Quantum Kinetic (Boltzmann) equation

- Describes rate of scattering into & out of states.

• Consider steady state solution $\partial_t D^A = \partial_t D^B = 0$

Use parameterisation: $D^B(\omega, k) = D^B(\omega, k)F(\omega) = F(\omega)D^A(\omega, k)$

• Rewrite equation as: $[D^B]^{-1} D^B [D^A]^{-1} = -[D^A]^k$

Separate $[D^B]^{-1} = i\partial_\omega - \vec{\nabla}^2/2m$

[See, e.g. Lifshitz and Pitaevskii *Physical Kinetics* Butterworth-Heinemann].

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$$[F, [D_0^R]^{-1}] = \Sigma^K - (\Sigma^R F - F \Sigma^A)$$

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$$[F, [D_0^R]^{-1}] = \Sigma^K - (\Sigma^R F - F \Sigma^A)$$

$$\frac{\partial F}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla F = I_{\text{in}} - I_{\text{out}}(F)$$

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Heisenberg-Langevin equations

- Consider photon-bath coupling: $H_{\text{int}} = \sum_{k,P} \zeta_{k,p} (\psi_k^\dagger \psi_p + \psi_p^\dagger \psi_k)$,

Heisenberg equation: $i\partial_t \psi_p = (\omega_p - i\gamma_p) \psi_p$

Solution: $\psi_p(t) = \psi_p(0) e^{-i\omega_p t} - iC_{p,k} \int dt' \psi_k(t') e^{-i\omega_k^2(t-t')}$

Substitute in: $i\partial_t \psi_k = \sum_P \zeta_{p,k} \psi_p + \dots$

In Markov approximation, $\sum_P \zeta_{p,k} e^{-i\omega_k^2(t-t')} = 2\alpha \delta(t-t')$

(cf bath retarded Green's function)

Gives result: $i\partial_t \psi_k = \omega_k \psi_k - i\gamma_k \psi_k +$

{Correlation length of the bath retarded Green's function}

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Density matrix equation of motion

- Markov approximation allows:

$$\frac{d}{dt}\rho_{\text{sys}} = -i [H_{\text{sys}}, \rho_{\text{sys}}] - i \text{Tr}_{\text{bath}} \{ [H_{\text{int}}, \rho_{\text{sys}} \otimes \rho_{\text{bath}}] \},$$

- » Different focus:
 - Non-equilibrium diagrams: Non-equal time one-particle correlations.
 - Many particles require extra diagrams.
 - Density matrix: Equal-time multiparticle correlations.
 - Multi-time correlations requires “Quantum regression theorem.”
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