Nonequilibrium quantum condensates: from microscopic theory to macroscopic phenomenology

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Acknowledgements

People:

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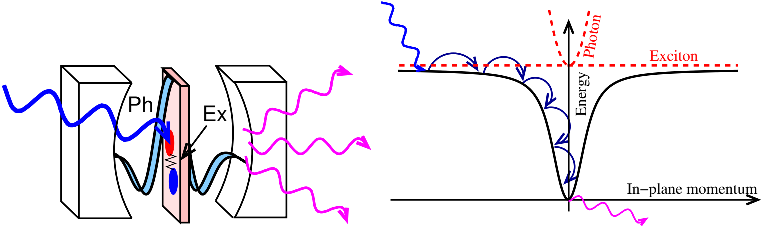
Non-equilibrium: Timescales



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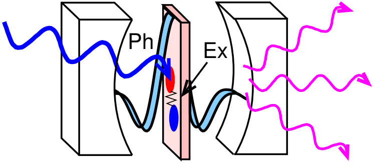

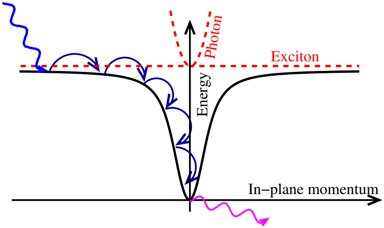

	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

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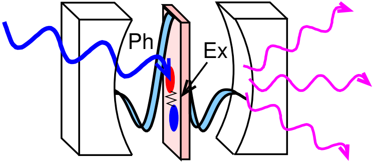

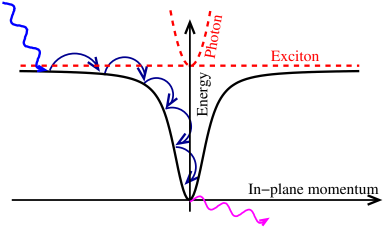

	Lifetime	Thermalisation	Linewidth	Temperature	
Atoms	10s	10ms	2.5×10^{-13}meV	10^{-8}K	10^{-9}meV
Excitons[a]	50ns	0.2ns	5×10^{-5}meV	1K	0.1meV
Polaritons	5ps	0.5ps	0.5meV	20K	2meV
Magnons[b]	1μs(??)	100ns(?)	2.5×10^{-6}meV	300K	30meV

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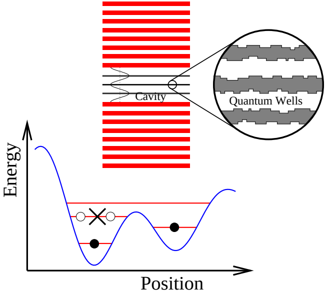
Overview

1. Microscopic non-equilibrium model
   - Model and mean-field theory
   - Fluctuations and stability of normal state
2. Macroscopic phenomenology
   - Gross Pitaevskii equation in an harmonic trap
     - Spontaneously rotating vortex lattice
   - Internal Josephson effect and spatial variation
     - Spin degree of fredom
     - Summary of two-mode model
     - Spin and spatial degrees of freedom

Polariton system model

Polariton model

- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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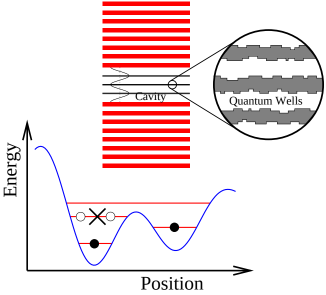
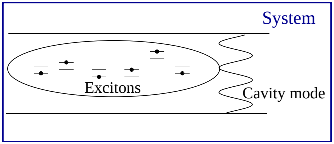
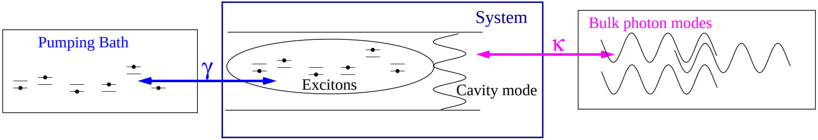


$$H_{sys} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{1}{\sqrt{\mathrm{Area}}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \mathrm{H.c.} \right]$$



Non-equilibrium model: baths



$$H = H_{\text{sys}} + H_{\text{sys,bath}} + H_{\text{bath}}$$

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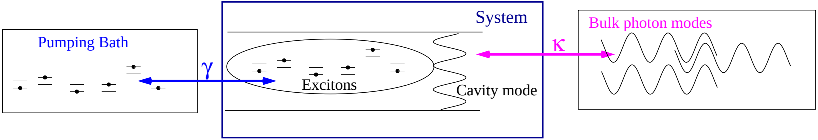

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Schematically: pump γ, decay κ

$$H_{\rm sys,bath} \simeq \sum_{\mathbf{p},\vec{k}} \sqrt{\kappa}\psi_{\mathbf{k}}\Psi^\dagger_{\mathbf{p}} + \sum_{\alpha,\beta} \sqrt{\gamma}\left(a^\dagger_\alpha A_\beta + b^\dagger_\alpha B_\beta\right) + \text{H.c.}$$

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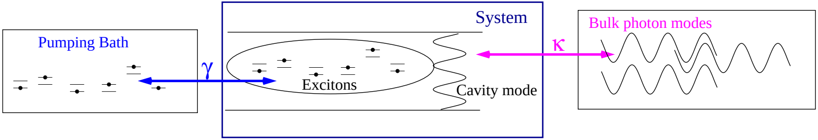

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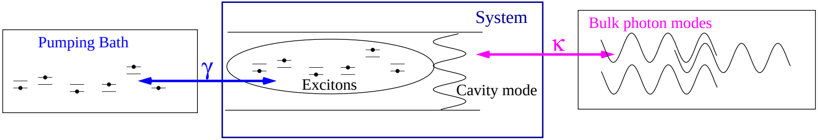

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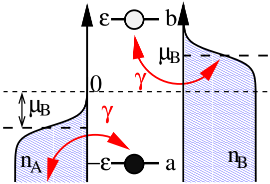
Ψ bath is empty. Pumping bath thermal, μ_B, T:



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Susceptibility:

$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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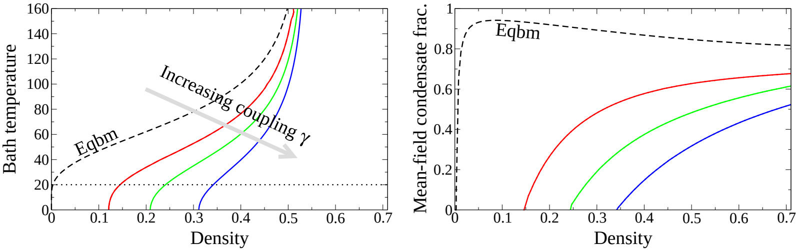

Fluctuations → Stability, Luminescence, Absorption

Keldysh approach:

$$\mathcal{D}_{R,A} = \mp i\theta[\pm(t-t')] \left\langle \left[\psi, \psi^\dagger\right]_- \right\rangle$$

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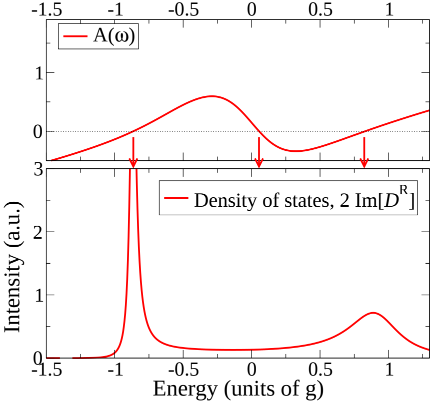

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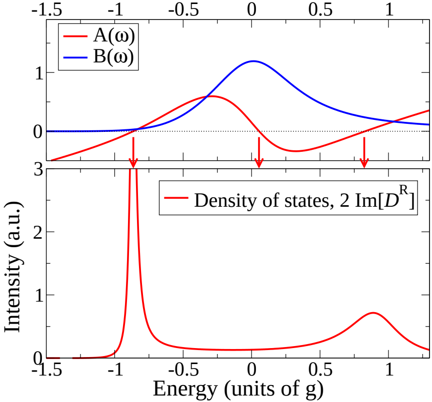

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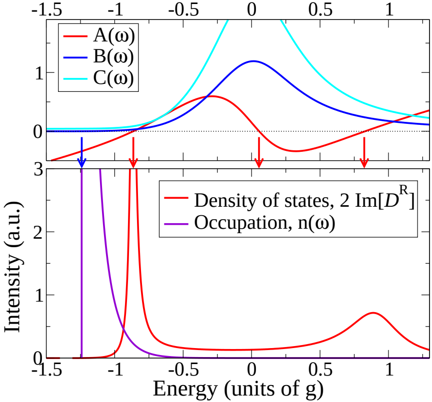

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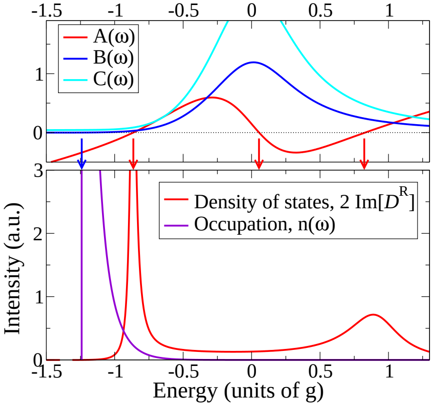
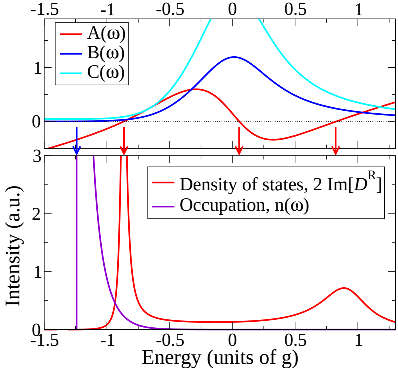


Linewidth, inverse Green's function and gap equation



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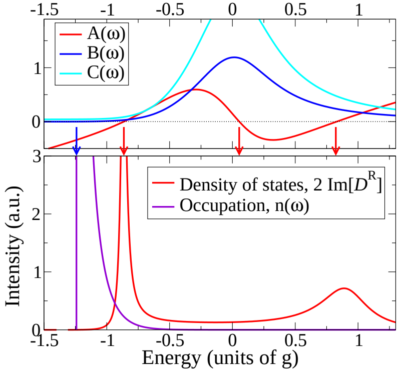
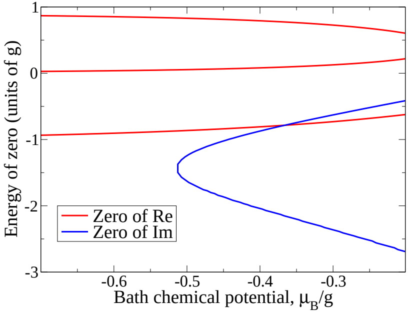
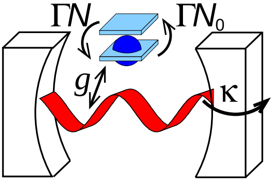




$[\mathcal{D}^R]^{-1}$ via Maxwell Bloch equations

Semiclassical EOM for: ψ, $P = n\langle -ia^\dagger b\rangle$, $N = n\langle b^\dagger b - a^\dagger a\rangle$



$$\partial_t \psi = -i\omega_k \psi - \kappa\psi + gP$$
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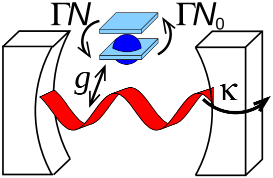

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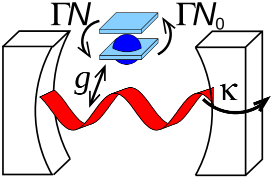

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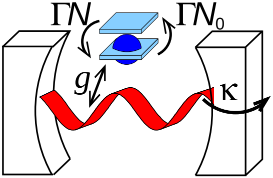

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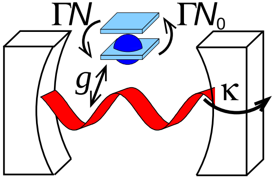

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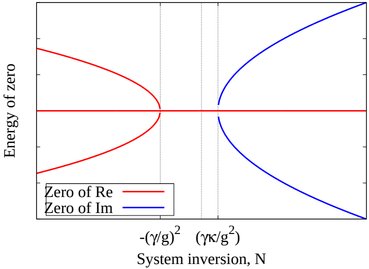

Overview

Limits of gap equation

Gap equation:

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- Local density limit:

Limits of gap equation

Gap equation:

$$(\mu_s - \omega_0 + i\kappa)\,\psi = \chi(\psi, \mu_s)\psi$$

- Local density limit: Gross-Pitaevskii equation

$$\left(i\hbar\partial_t + i\kappa - \left[V(r) - \frac{\hbar^2\nabla^2}{2m}\right]\right)\psi(r) = \chi(\psi(r,t))\psi(r,t)$$

Nonlinear, complex susceptibility $\chi(\psi(r,t))$

Limits of gap equation

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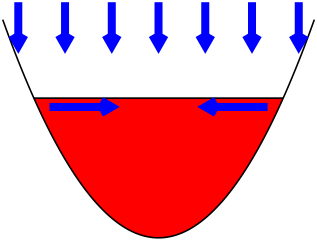

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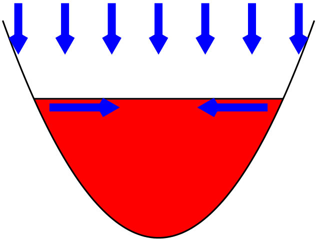

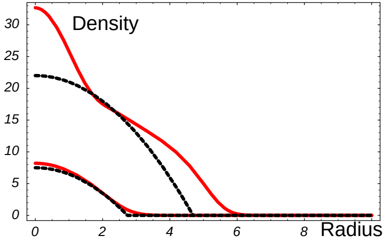

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Stability of Thomas-Fermi solution

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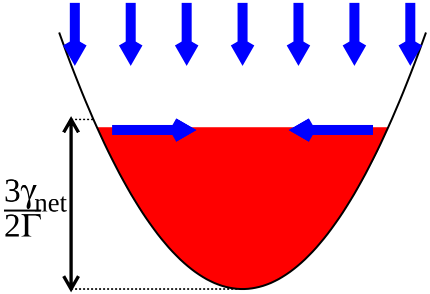

Stability of Thomas-Fermi solution

High m modes: $\delta\rho_{n,m} \simeq e^{im\theta} r^m \dots$

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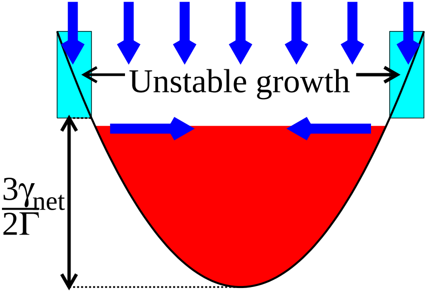

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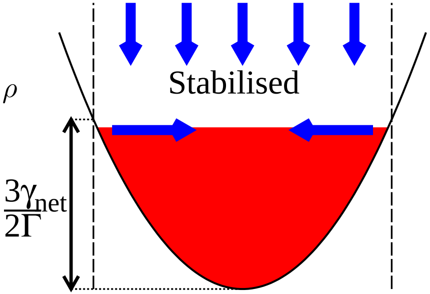

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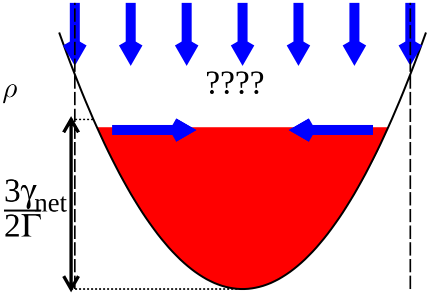
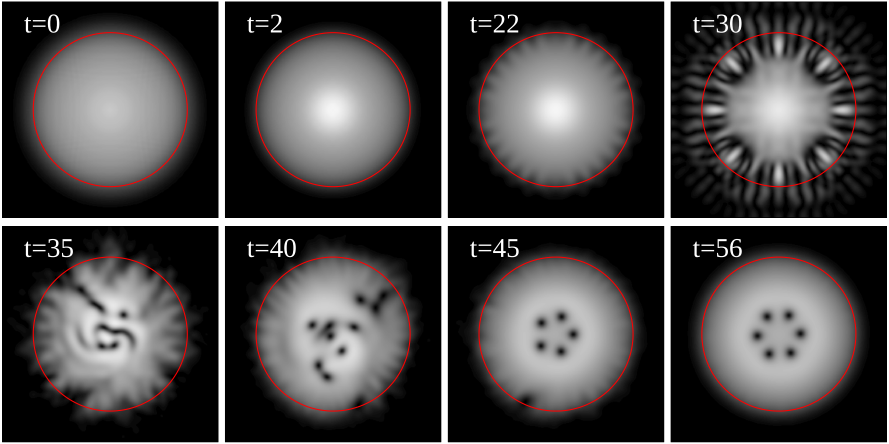


Time evolution:



[Keeling & Berloff, PRL, '08]

Why vortices

Rotating solution: $i\partial_t \psi = (\mu - 2\Omega L_z)\psi$

$$\mathbf{v} = \Omega \times \mathbf{r}, \quad \Omega = \omega, \quad \rho = \frac{\mu}{U} = \frac{\gamma_{\text{net}}}{\Gamma}\Theta(r_0 - r)$$

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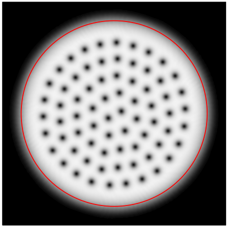

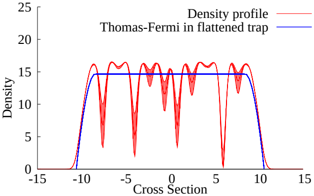

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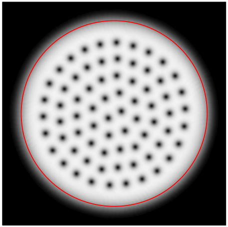

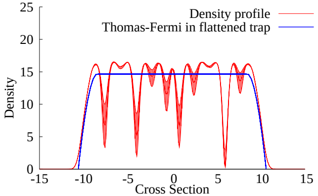

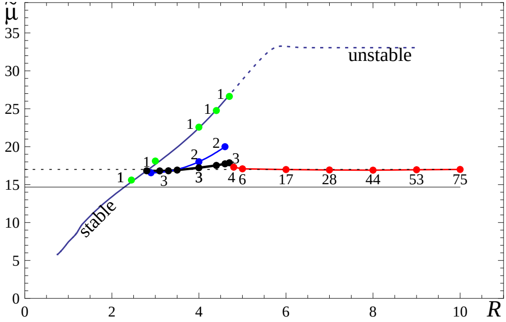

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Overview

Polariton spin degree of freedom

- Results so far do not involve polariton spin:
  Left- and Right-circular polarised polaritons states.
- For weakly-interacting dilute Bose gas model:
- Tendency to biexciton formation $\rightarrow U_1$. Magnetic field: Δ
- J_2 Circular Symmetry $\rightarrow D_{2d}$ — two equivalent axes.
  J_1 $D_{2d} \rightarrow C_{2v}$ — inequivalent axes.

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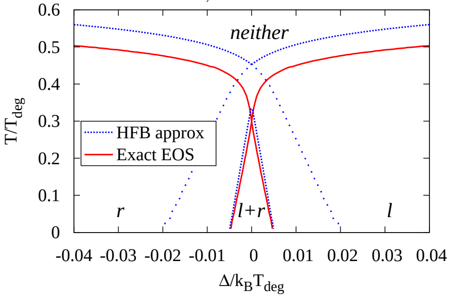
$J_1 = J_2 = 0$.

For $U_1 = 0.5$, $\Psi_{L,R}$ decouple.



Circular $\rightarrow$ Eliptical transitions.

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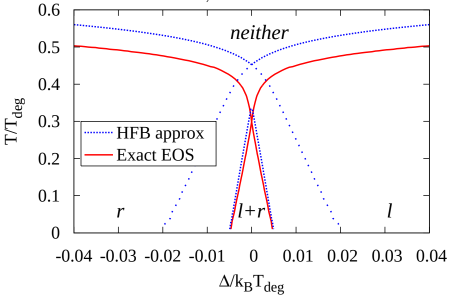

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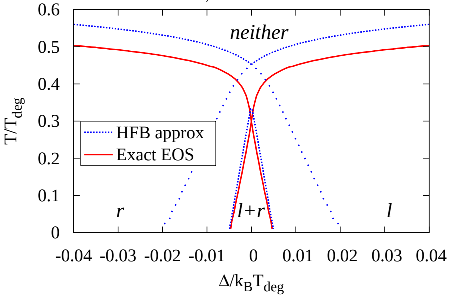

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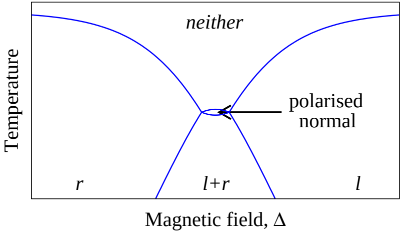

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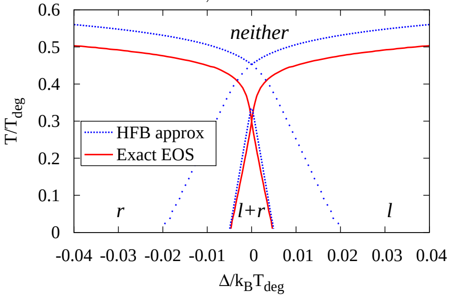

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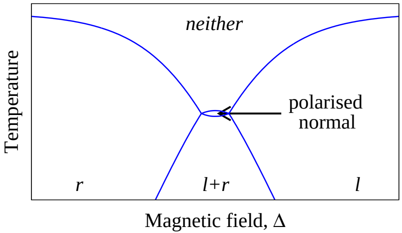

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Non-equilibrium spinor system

Spinor Gross-Pitaevskii equation:

$$i\partial_t \psi_L = \Big[-\frac{\hbar^2 \nabla^2}{2m} + V(r) - \frac{\Delta}{2} + U_0|\psi_L|^2 + (U_0 - 2U_1)|\psi_R|^2 + i\left(\gamma_{\text{eff}}(\mu_B) - \kappa - \Gamma|\psi_L|^2\right)\Big]\psi_L + J_1\psi_R$$

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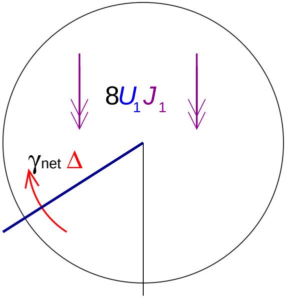

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Damped, driven pendulum

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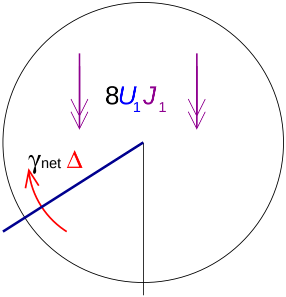

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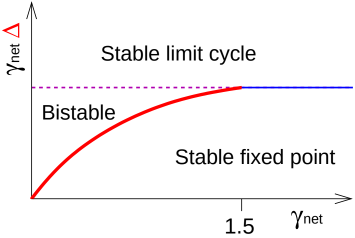

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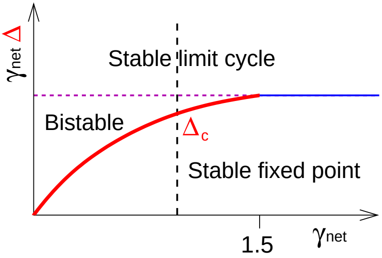
[e.g. Strogatz, Nonlinear dynamics and chaos]

Two-mode model bistability



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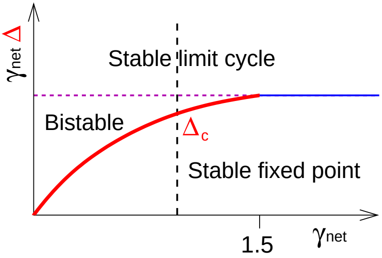

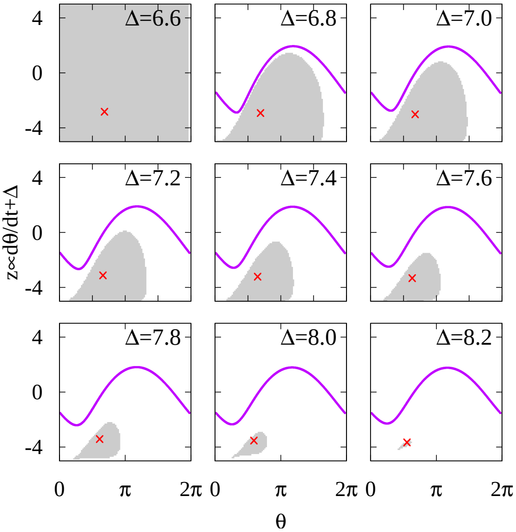

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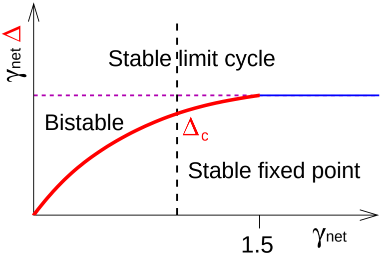
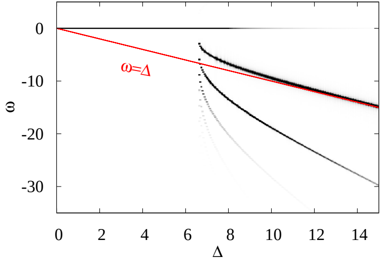
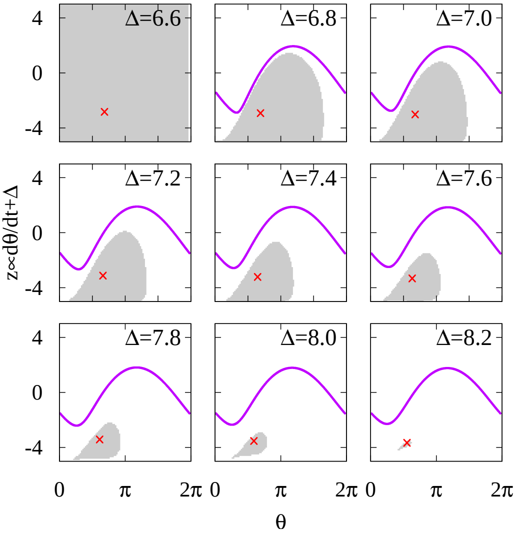




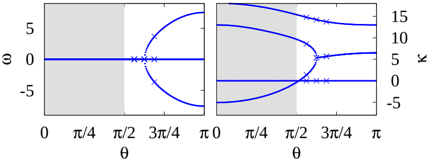
$\Delta \gtrsim \Delta_c$: $\omega \simeq [\ln(\Delta - \Delta_c)]^{-1}$



Homogenous case: stability at $\Delta < \Delta_c$

Damped oscillations
$\Omega_p^2 = -8 U_1 J_1 R_0 \cos(\theta)$

$$\omega - i\kappa \simeq \begin{cases} 0 \\ -i\gamma_{\text{net}} \pm i\sqrt{\gamma_{\text{net}} - \Omega_p^2} \\ -2i\gamma_{\text{net}} \end{cases}$$



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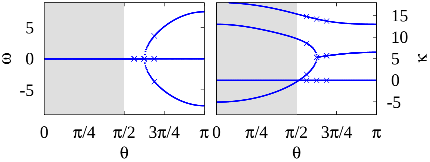

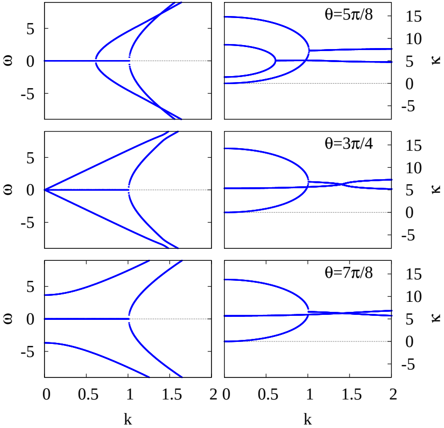

If $\Omega_p^2 = \gamma_{\text{net}}$ degenerate modes:
$\omega \propto k$ for spin wave.

Trapped spinor system

Consider: $V(r) = m\omega^2 r^2/2$ $\gamma_{\text{net}} \rightarrow \gamma_{\text{net}}\Theta(r_0 - r)$

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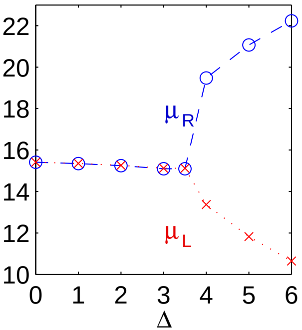

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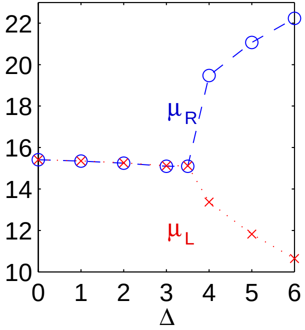

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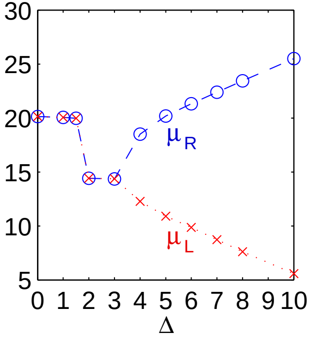

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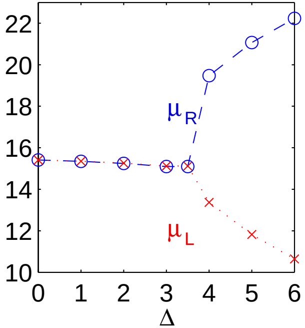

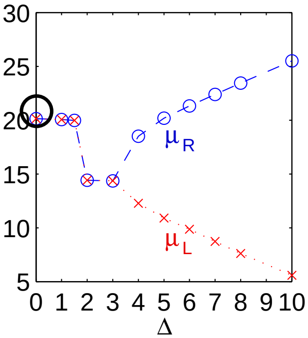

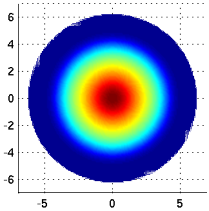

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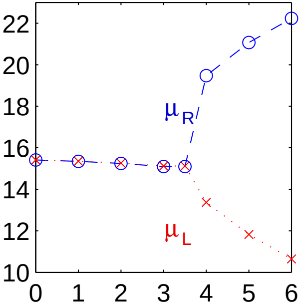

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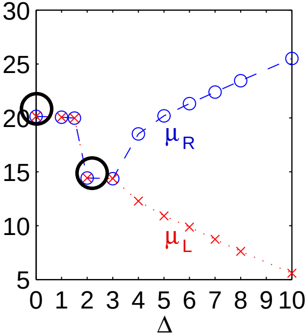
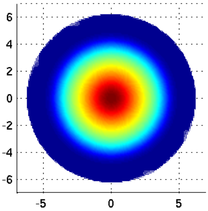
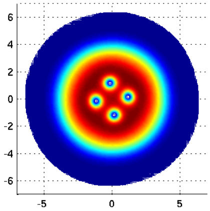
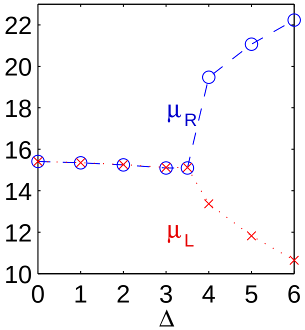






Trapped spinor system — phase portraits

"Simple" case not so simple



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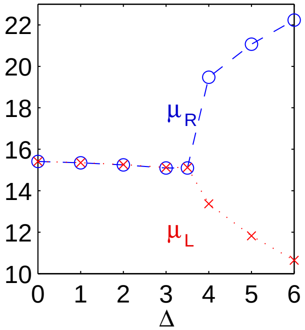

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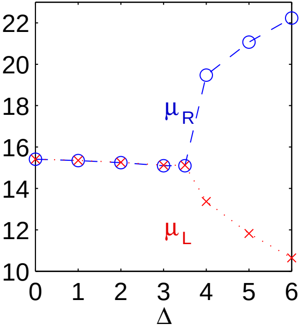

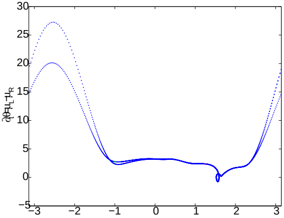

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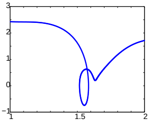

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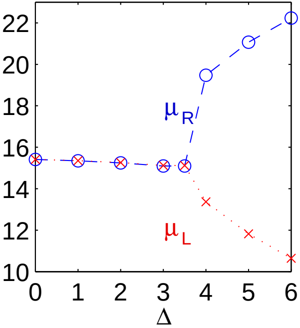

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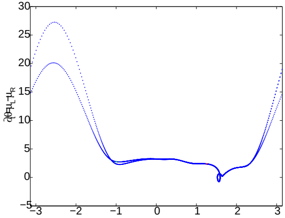

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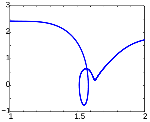

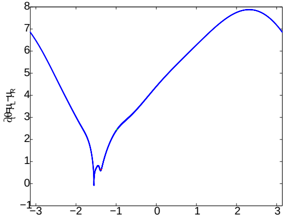

$\Delta = 3.25$

Spatial variation

Varieties of behaviour possible as $\theta(\mathbf{r})$, not $\bar{\theta}$ needed to define state.

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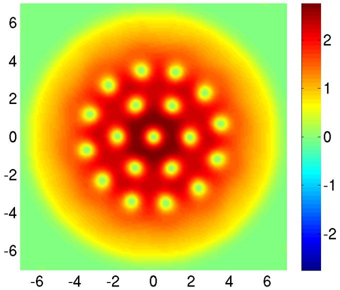

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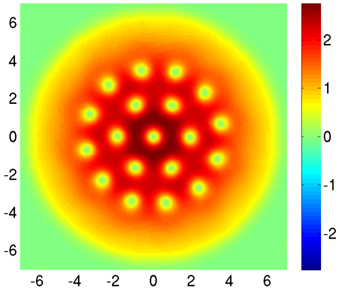

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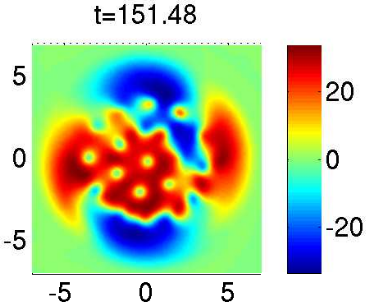
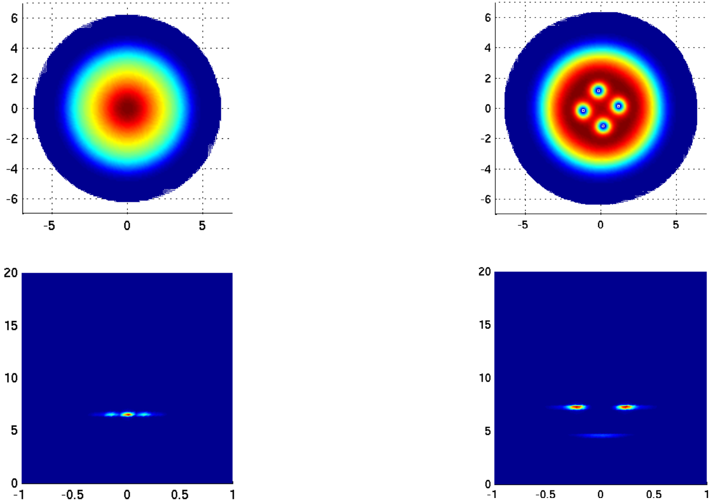


$J_1 = 1; r_0 > r_{TF}; \Delta = 6$
Counter-rotating.

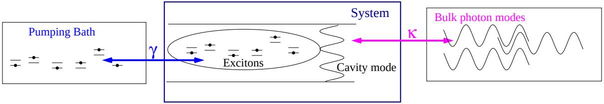
Vortex lattices, desynchronisation and spectrum

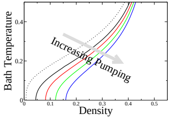
Vortex lattice → phase gradients; finite k, ω.



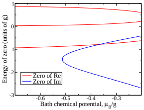
Conclusions

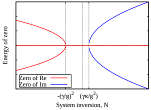
- Effects of pumping on mean-field theory



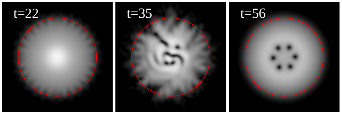


- Instability of normal state
  Translating: condensation ↔ lasing

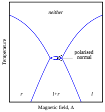


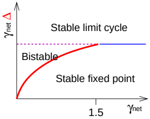


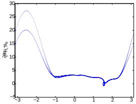
- Modification to Thomas-Fermi profile
  Spontaneous rotating vortex lattice



- Spinor model.
  Steady states & fluctuations.







Acknowledgements

People:

Funding:

EPSRC Engineering and Physical Sciences Research Council

Pembroke College

Extra slides

Blueshfit and experimental phase boundary

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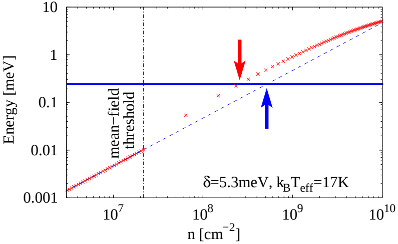

Clean limit:

$$\delta E_{\mathrm{LP}} \simeq \mathcal{R}y_{\mathrm{X}} a_{\mathrm{X}}^2 n + \Omega_{\mathrm{R}} a_{\mathrm{X}}^2 n$$

Here: $\Omega_{\mathrm{R}} a_{\mathrm{X}}^2 \to \Omega_{\mathrm{R}} \xi^2$
[PRB 77 235313]

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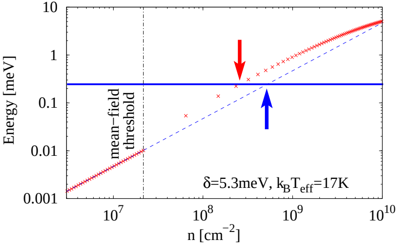

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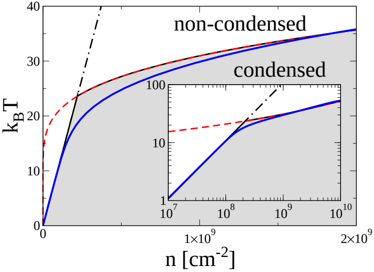
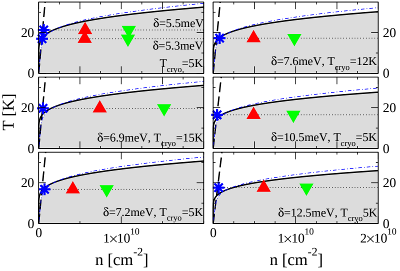


CdTe:



Limits of gap equation

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$$\mu_s - \omega_0 + i\kappa = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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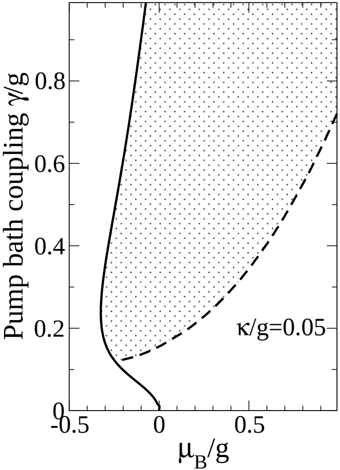
$$\omega_0 - \mu_s = \frac{g^2}{2E} \tanh\left(\frac{\beta E}{2}\right)$$

Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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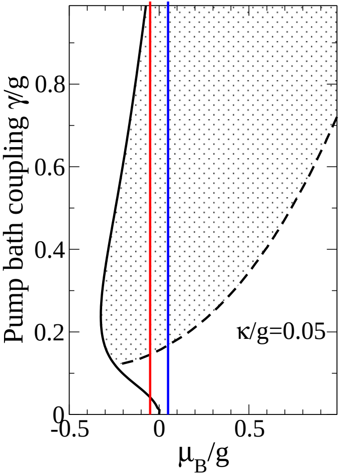

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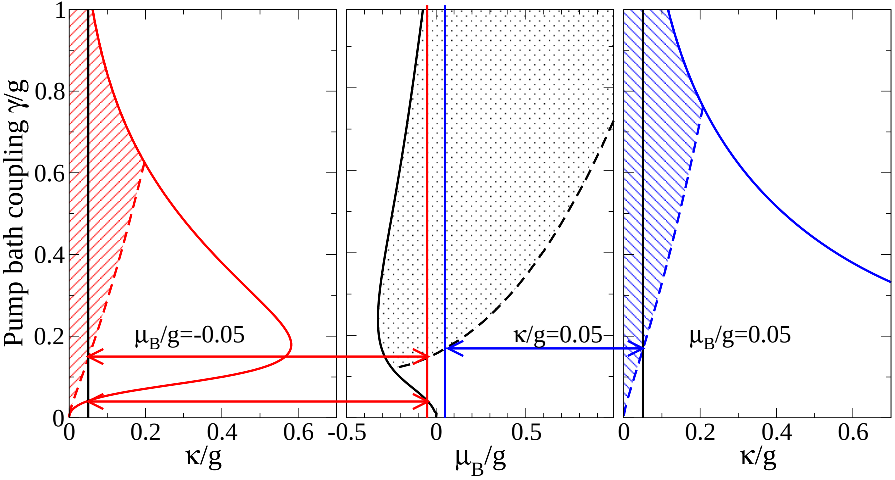


Overview

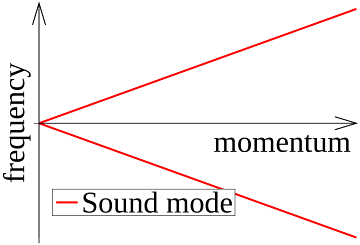
Fluctuations above transition

When condensed

Det $\left[\mathcal{G}_R^{-1}(\omega, \mathbf{k})\right] = \omega^2 - c^2\mathbf{k}^2$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

Fluctuations above transition

When condensed

$$\mathrm{Det}\left[\mathcal{G}_R^{-1}(\omega,\mathbf{k})\right] = (\omega+ix)^2+x^2-c^2\mathbf{k}^2$$

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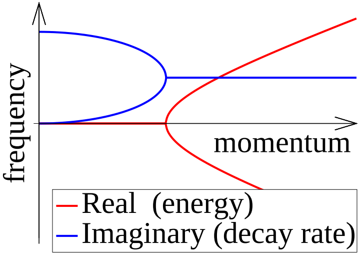
$$\omega^* = -ix \pm \sqrt{c^2\mathbf{k}^2 - x^2}$$



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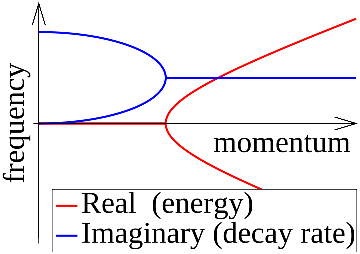

Correlations (in 2D): $\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$

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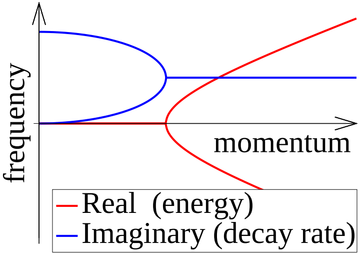

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$$\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta\begin{cases}\ln(r/\xi) & r\to\infty, t\simeq 0\\ \frac{1}{2}\ln(c^2t/x\xi^2) & r\simeq, t\to\infty\end{cases}\right]$$

[Szymańska et al., PRL '06; PRB '07]

Finite size effects: Single mode vs many mode

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$\mathcal{D}_{\phi\phi}(t,r,r')$ from sum of phase modes. Study $ct \gg r$ limit:

$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_n^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{|(\omega+ix)^2+x^2-(n\Delta)^2|^2}$$

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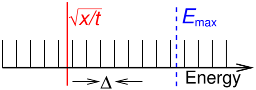
$\Delta \ll \sqrt{x/t} \ll E_{\rm max}$



$\mathcal{D}_{\phi\phi} \sim 1 + \ln(E_{\rm max}\sqrt{t/x})$

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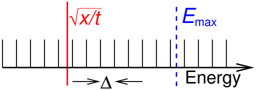

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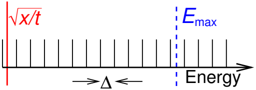

$\mathcal{D}_{\phi\phi} \sim \left(\frac{\pi C}{2x}\right)\left(\frac{t}{2x}\right)$

Relating finite-size spectrum to self phase modulation

Single mode spectrum:

$$\mathcal{D}_{\phi\phi}(t, r, r) \propto \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1 - e^{i\omega t})}{|(\omega + ix)^2 + x^2 - (n\Delta)^2|^2}$$

Connection to Kubo Formula [Eastham and Whittaker, arXiv:0811.4333]

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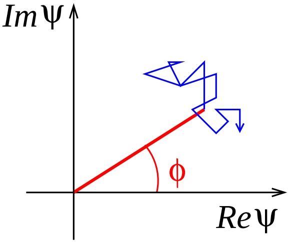

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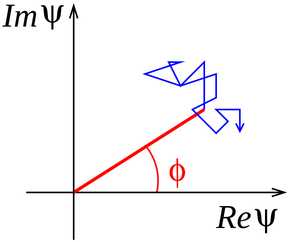

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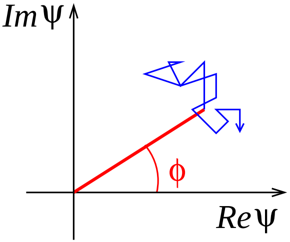

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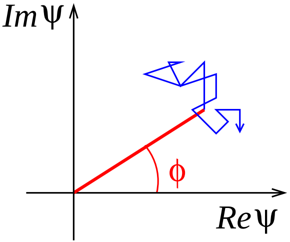

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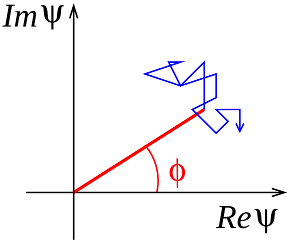

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Instability of Thomas-Fermi: details

$\psi = \sqrt{\rho} e^{i\phi} \quad \mathbf{v} = \frac{\hbar}{m} \nabla \phi$

$$\frac{1}{2}\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = (\gamma_{\text{net}} - \Gamma \rho)\rho$$

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If $\gamma_{\mathrm{net}}, \Gamma \to 0$, can find normal modes in 2D trap:

$$\delta\rho_{n,m}(r,\theta,t) = e^{im\theta} h_{n,m}(r) e^{i\omega_{n,m}t}$$

$$\omega_{n,m} = \omega 2\sqrt{m(1+2n) + 2n(n+1)}$$

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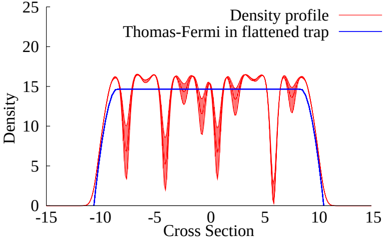
Consider $\rho \to \rho + \delta\rho, \mathbf{v} \to \mathbf{v} + \delta\mathbf{v}$.

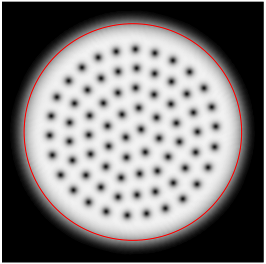
Add weak pumping/decay:

$$\omega_{n,n} \to \omega_{n,m} + i\gamma_{\text{net}} \left[\frac{m(1+2n) + 2n(n+1) - m^2}{2m(1+2n) + 4n(n+1) + m^2}\right]$$

Instability

Why vortices





$$\nabla\cdot[\rho(\mathbf{v}-\Omega\times\mathbf{r})] = (\gamma_{\mathrm{net}}\Theta(r_0-r)-\Gamma\rho)\,\rho,$$

$$\mu = \frac{m}{2}|\mathbf{v}-\Omega\times\mathbf{r}|^2 + \frac{m}{2}r^2(\omega^2-\Omega^2)+U\rho-\frac{\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}$$

$$\mathbf{v}=\Omega\times\mathbf{r},\quad \Omega=\omega,\quad \rho=\frac{\gamma_{\mathrm{net}}}{\Gamma}\Theta(r_0-r)=\frac{\mu}{U}$$

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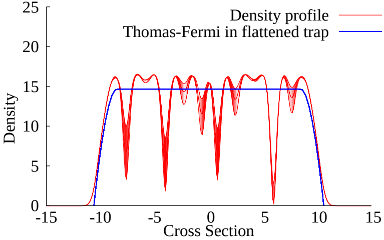

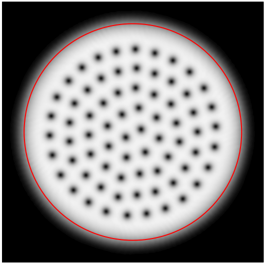

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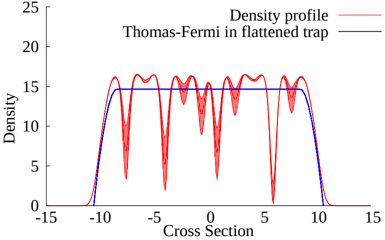

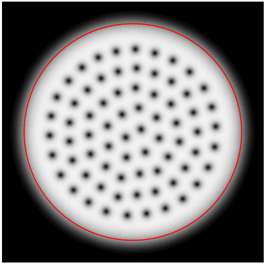

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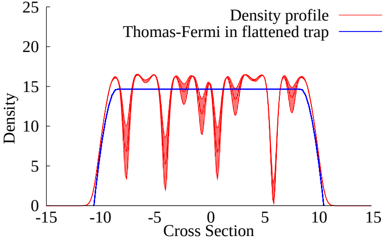

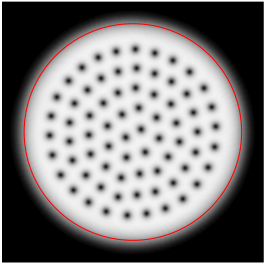

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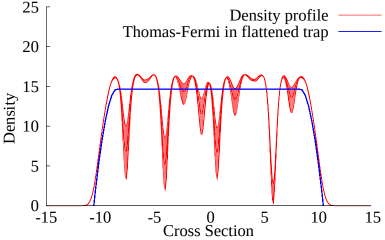

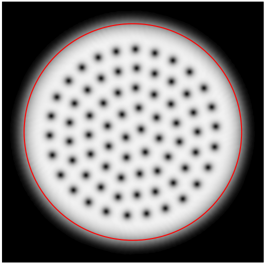

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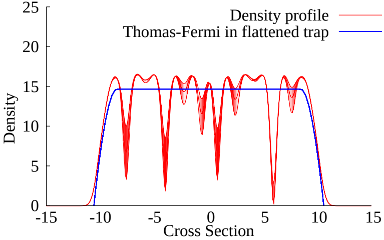

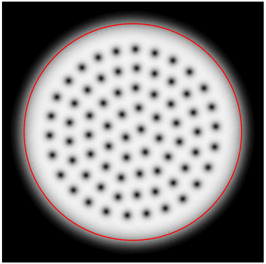

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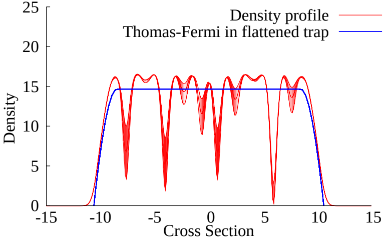

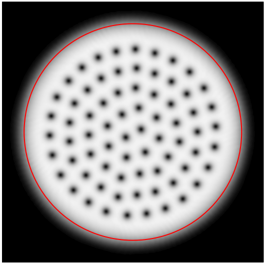

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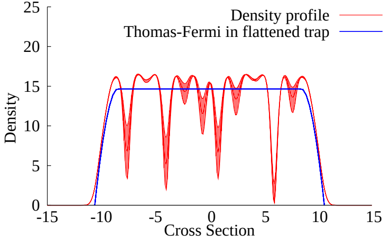
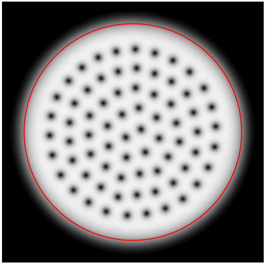
$$\mathbf{v} = \Omega \times \mathbf{r}, \quad \Omega = \omega, \quad \rho = \frac{\gamma_{\text{net}}}{\Gamma}\Theta(r_0 - r) = \frac{\mu}{U}$$

Why vortices





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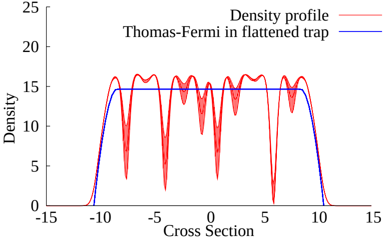

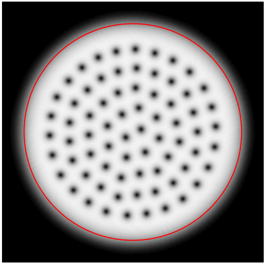

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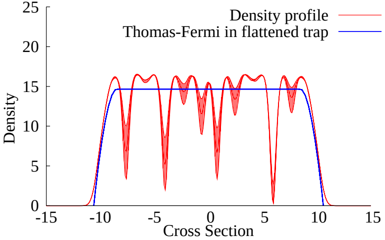

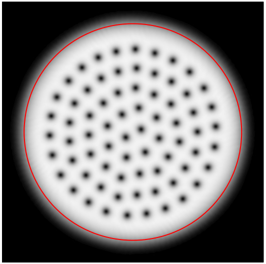

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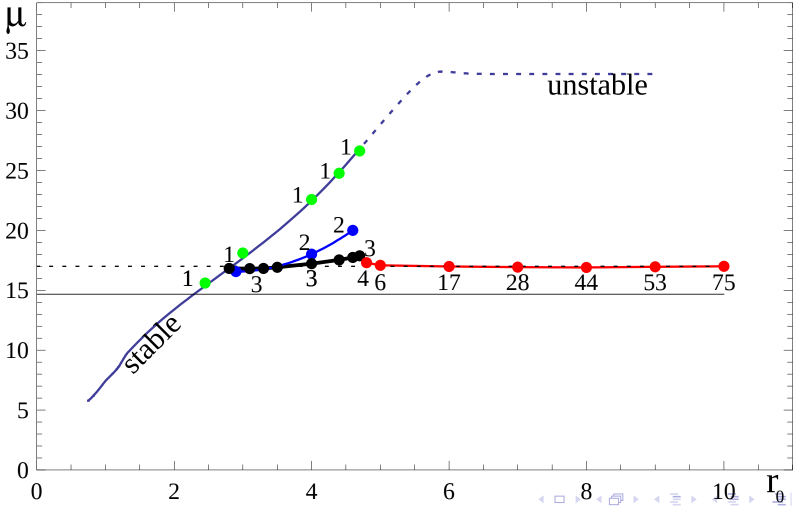
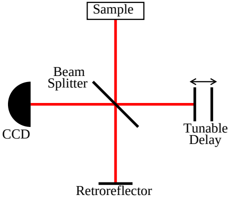
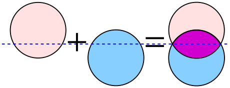
Why vortices: chemical potential vs size

Thomas-Fermi : $\mu \propto r_0^2$ Vortex : $\mu = \frac{U\gamma_{\text{net}}}{\Gamma}$



Observing vortices: fringe pattern





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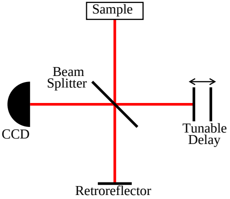

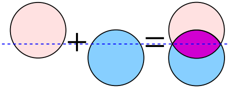

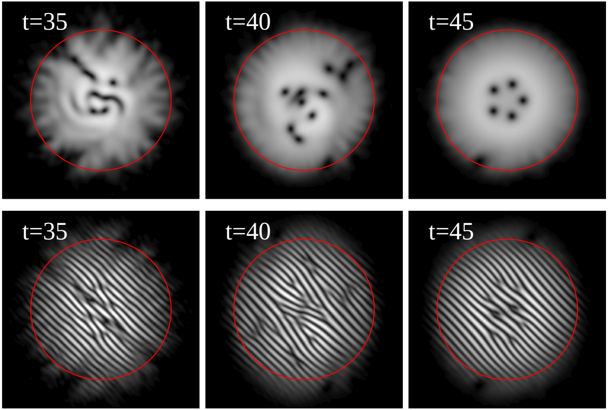

Spin in terms of ~~two~~four-level systems

To include spin, replace 2 level system with 4 levels: $|0\rangle, |L\rangle, |R|\rangle, |LR\rangle$

- Bi-exciton binding $E_{XX} \leftrightarrow U_1$
- Mean-field: find polarisation given ψ_L, ψ_R.
- E_{XX} has weak effect on T_c

[Marchetti *et al* PRB, '08]

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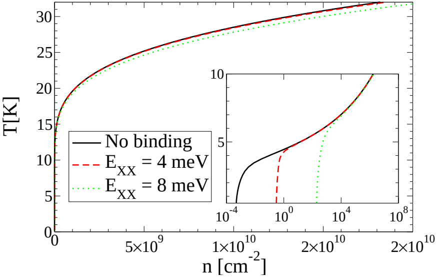

[Marchetti *et al* PRB, '08]

Mathematical outline

- 2D Single component equation of state: $n(\mu, T) = Tf(x = \mu/T)$
- For two components:

$$n_0 = T\left[f\left(\frac{\mu+\Omega}{T}\right) + f\left(\frac{\mu-\Omega}{T}\right)\right]$$

- At critical point for one component:

$$n_0 = T\left[f_c + f\left(x_c + \frac{2\Omega}{T}\right)\right]$$

- Hence:

$$T = \frac{n_0}{f_c + f\left(x_c + \frac{2\Omega}{T}\right)}$$

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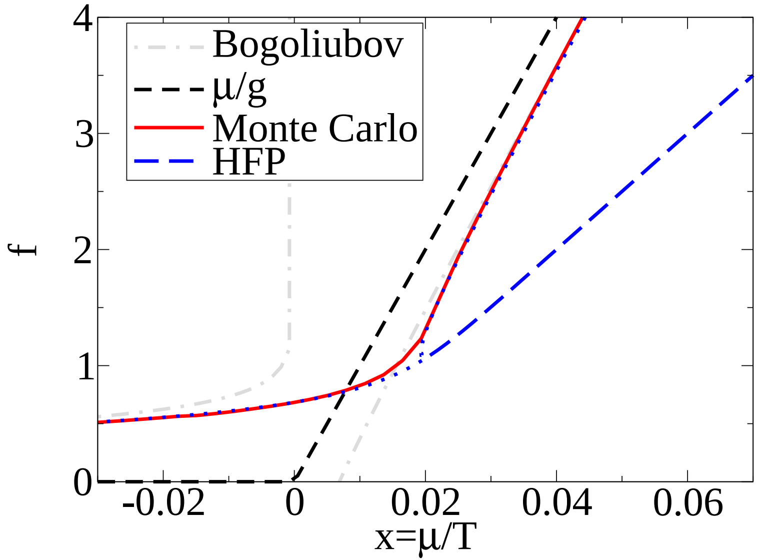
Graphical implementation of $T = n_0 / \left[f_c + f\left(x_c + \frac{2\Omega}{T} \right) \right]$



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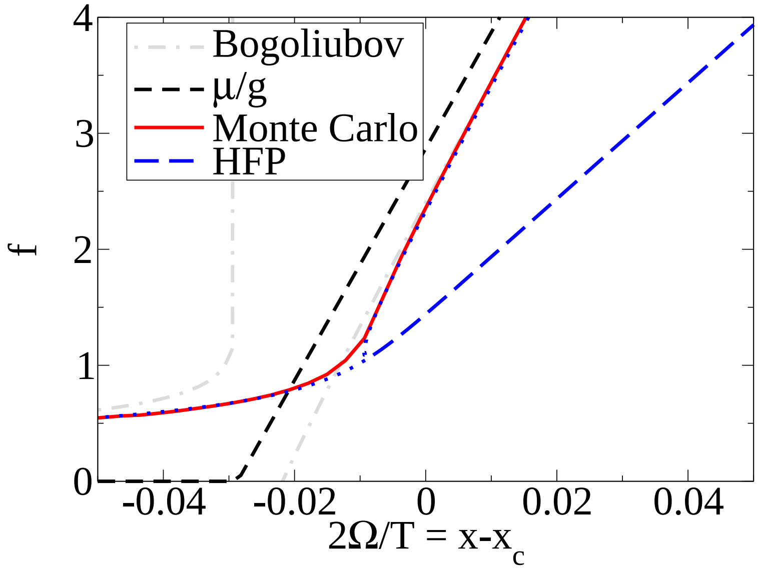

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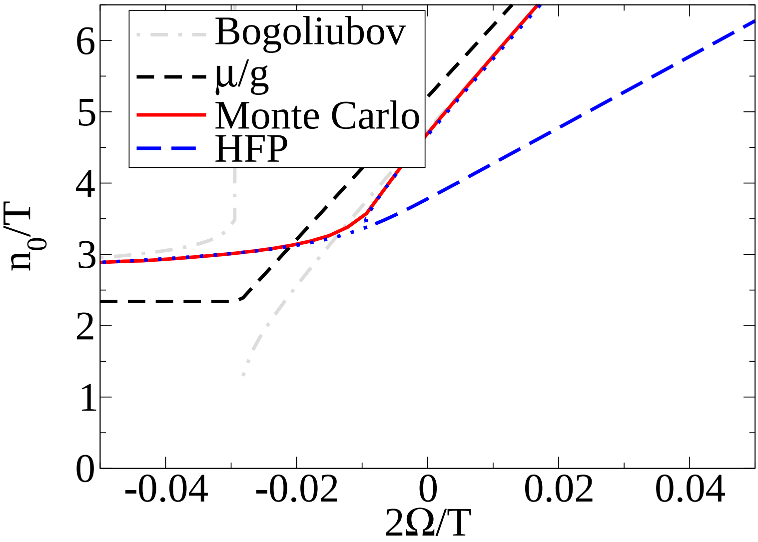

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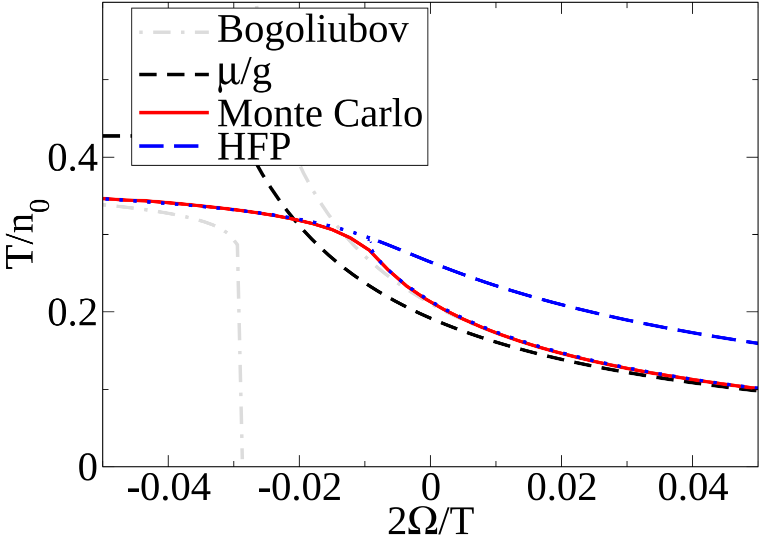

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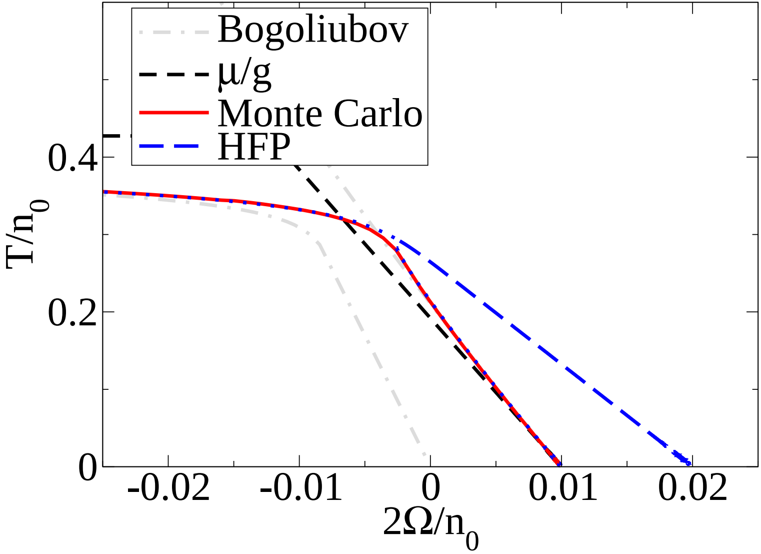

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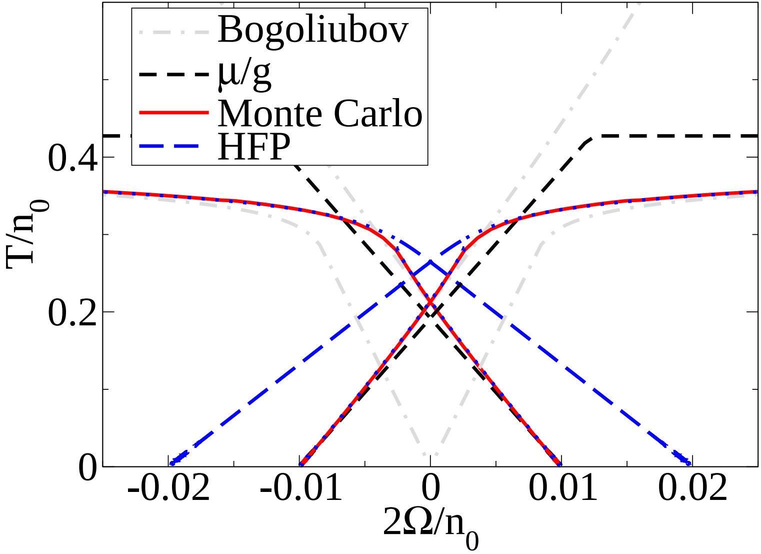

Spatial freedom: Homgeneous case $\Delta < \Delta_c$

$$\ddot{\theta} + 2\gamma_{\text{net}}\dot{\theta} = 8 U_1 J_1 R_0 \sin(\theta) - 2\gamma_{\text{net}}\Delta$$

- Steady state condition: $8 U_1 J_1 R_0 \sin(\theta) = 2\gamma_{\text{net}}\Delta$
- $\psi_{L,R} \to e^{-i\mu t}\left(\psi^0_{L,R} + u_1 e^{-i\mathbf{k}\cdot\mathbf{r}+(-i\omega-\kappa)t} + v_1^* e^{i\mathbf{k}\cdot\mathbf{r}+(i\omega-\kappa)t}\right)$
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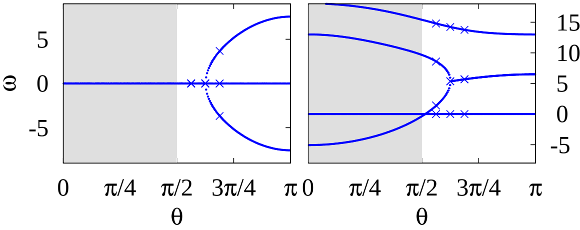
$$\omega - i\kappa = 0, -2i\gamma_{\text{net}}$$
$$-i\gamma_{\text{net}} \pm i\sqrt{\gamma_{\text{net}} - \Omega_p^2}$$



Stability requires $\Omega_p^2 > 0$. If $\Omega_p^2 < \gamma_{\text{net}}$ overdamped.

Superfluidity

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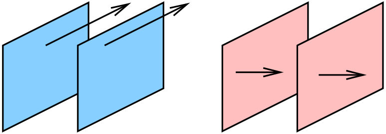

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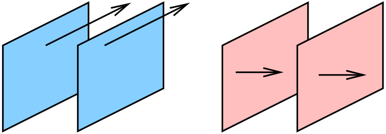

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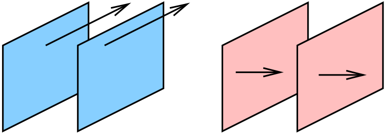

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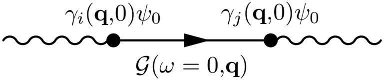
$\Delta\chi_{ij}(q) =$ 

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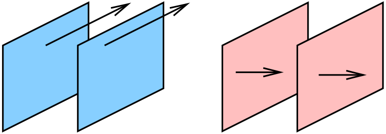

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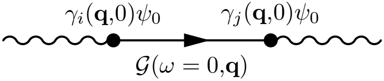
$\Delta\chi_{ij}(q) =$  $+ \ldots$

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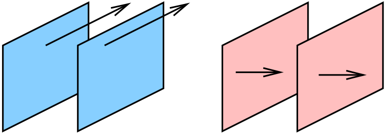

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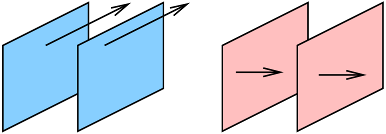

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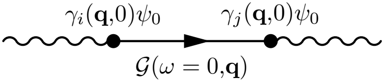
Superfluid part,

$\rho_s \propto \lim_{q\to 0}(\chi_L - \chi_T)$.

$J_i(q) = \langle\psi^\dagger\gamma_i(q)\psi\rangle$

Static ρ_S survies

$$\Delta\chi_{ij}(q) =$$



$$+ \ldots$$

$$= \gamma_i(q)\mathcal{G}_R(\omega = 0, \vec{q})\gamma_j(q) + \ldots$$

$$\propto \frac{q_i\,(\ldots)\,q_j}{(\omega + ix)^2 + x^2 - c^2\mathbf{q}^2} + \ldots$$