Nonequilibrium quantum condensates: from microscopic theory to macroscopic phenomenology

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Quantum Aggregates, April 2009

Acknowledgements

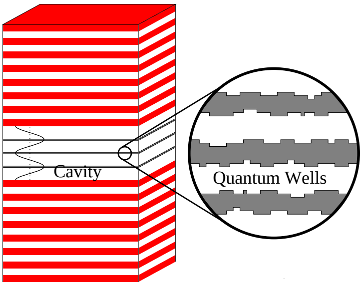
People:

Funding:

EPSRC Engineering and Physical Sciences Research Council

Pembroke College

Microcavity Polaritons



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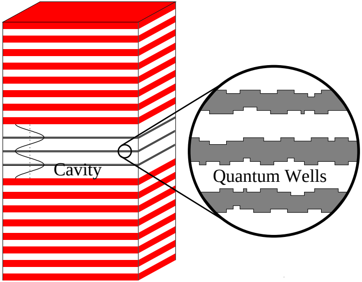

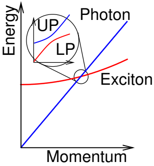

[Pekar, JETP(1958)]
[Hopfield, Phys. Rev.(1958)]

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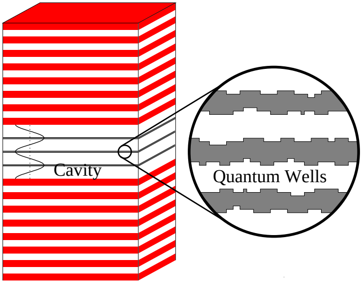

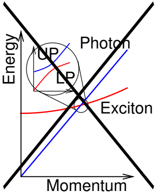

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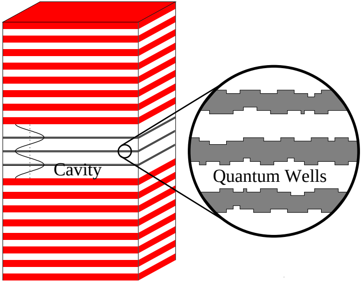

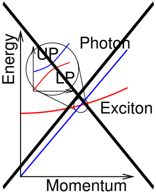

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Cavity photons:

$$\omega_k = \sqrt{\omega_0^2 + c^2 k^2}$$
$$\simeq \omega_0 + k^2/2m^*$$
$$m^* \sim 10^{-4} m_e$$

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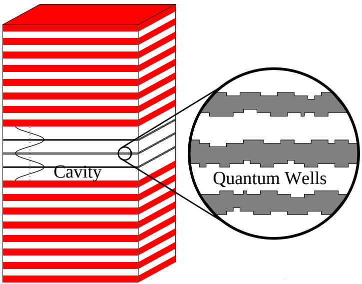

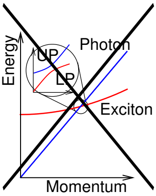

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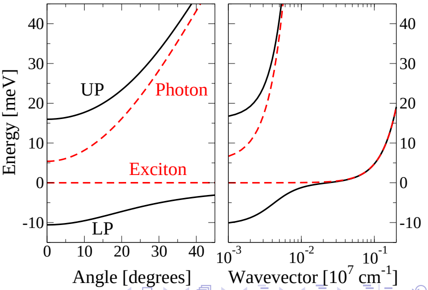
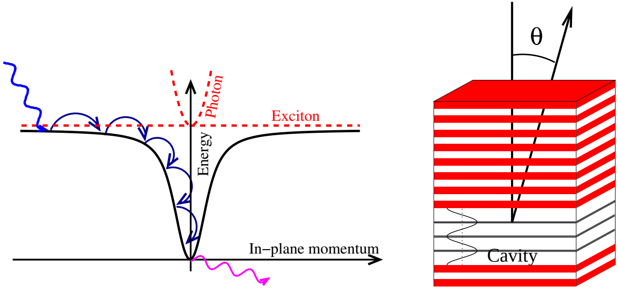
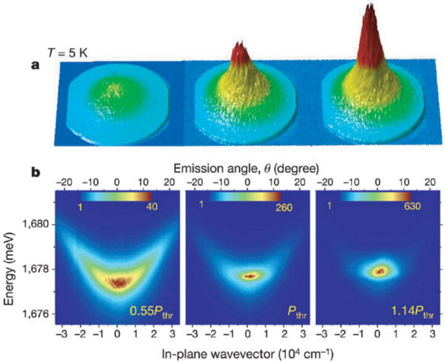


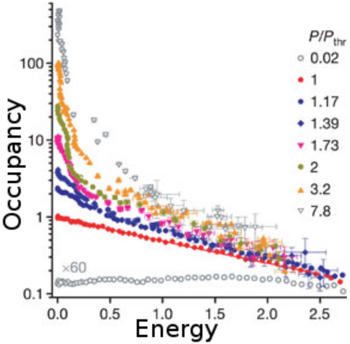
Non-equilibrium system



Polariton experiments: Momentum/Energy distribution

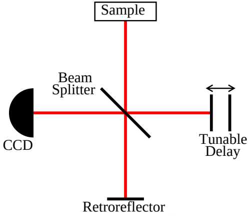


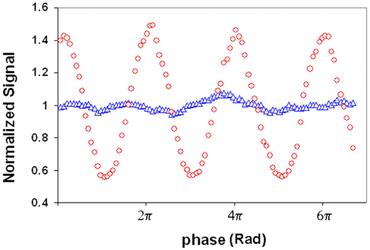
[Kasprzak, et al., Nature, 2006]



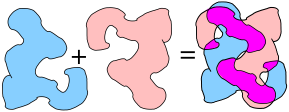
Polariton experiments: Coherence

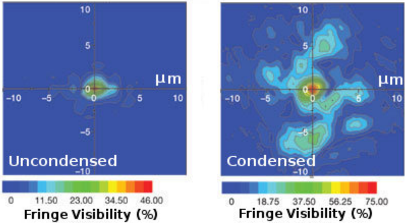
Basic idea:





Coherence map:

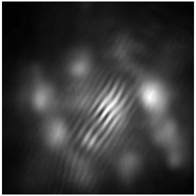


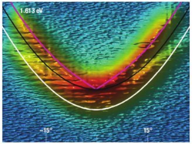


[Kasprzak, et al., Nature, 2006]

Other polariton condensation experiments

- Old measurements of $\langle N(t)N(t+\tau)\rangle$
  [Deng *et al* PNAS 100 15318 (2003)]
- Stress traps for polaritons
  [Balili *et al* Science 316 1007 (2007)]
- Temporal coherence and line narrowing
  [Love *et al* Phys. Rev. Lett. 101 067404 (2008)]
- Quantised vortices in disorder potential
  [Lagoudakis *et al* Nature Phys. 4, 706 (2008)]
- Changes to excitation spectrum
  [Utsunomiya *et al* Nature Phys. 4 700 (2008)]
- Soliton propagation
  [Amo *et al* Nature 457 291 (2009)]

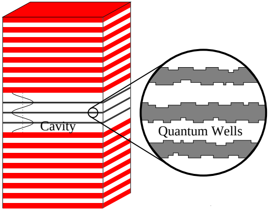




Overview

1. Introduction to microcavity polaritons
2. Model and review of equilibrium results
   - Disorder-localised exciton model
   - Equilibrium mean field theory
3. Non-equilibrium model and mean-field theory
   - Meaning of mean-field condition
4. Macroscopic phenomenology
   - Gross Pitaevskii equation in an harmonic trap
5. Fluctuations and correlations
   - Fluctuations about mean-field theory

Excitons in a disorderd Quantum well



Exciton states in disorder:

$$\left[-\frac{\nabla^2_{\mathbf{R}}}{2m_X} + V(\mathbf{R})\right]\Phi_\alpha(\mathbf{R}) = \varepsilon_\alpha \Phi_\alpha(\mathbf{R})$$

$V(\vec{R})$ smoothed by exciton Bohr radius

[PRL 96 066405 (2006); PRB 76 115326 (2007)]

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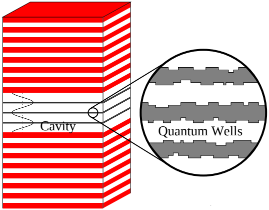

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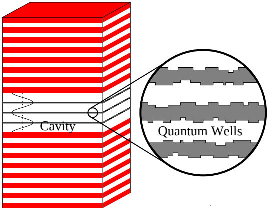

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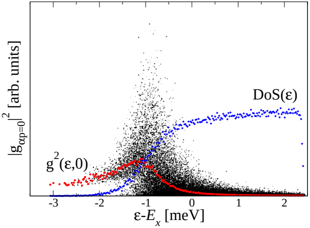

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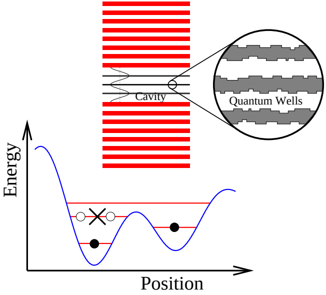
Polariton system model

Polariton model

- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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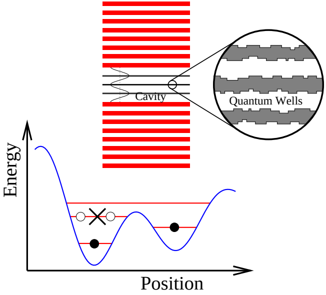
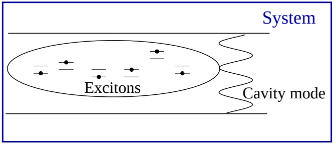


$$H_{\mathrm{sys}} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{1}{\sqrt{\mathrm{Area}}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \mathrm{H.c.} \right]$$



Equilibrium: Mean-field theory

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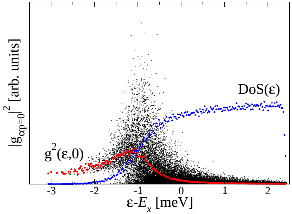

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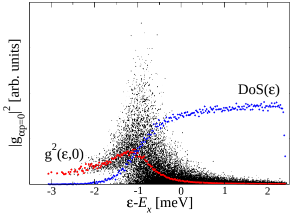
Mean-field theory:
Self-consistent polarisation and field

$$\left[i\partial_t + \omega_0 - \frac{\nabla^2}{2m} \right] \psi = \frac{1}{\sqrt{A}} \sum_{\alpha} g_{\alpha} P_{\alpha}$$



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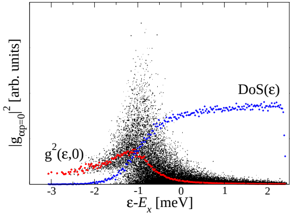

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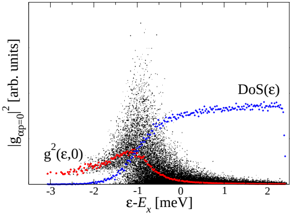

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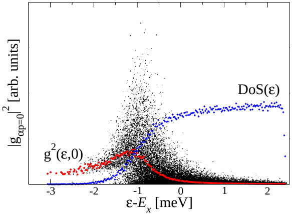

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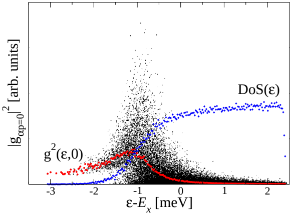

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Density

$$\rho = |\psi|^2 + \frac{1}{A} \sum_\alpha \left[\frac{1}{2} - \frac{\tilde{\epsilon}_\alpha}{4E_\alpha} \tanh(\beta E_\alpha) \right]$$

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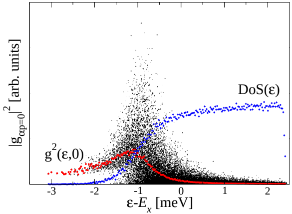

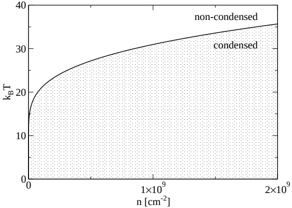

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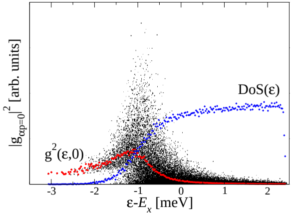

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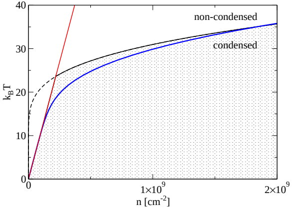
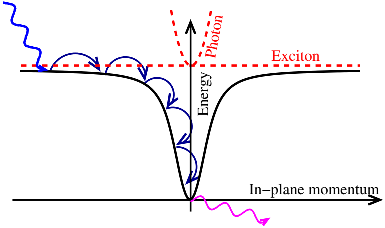
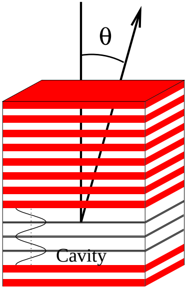


Overview

Non-equilibrium system





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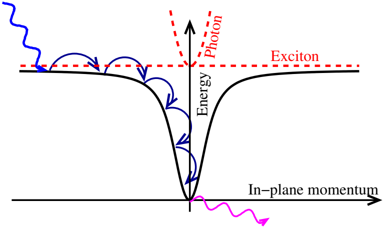

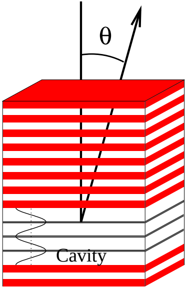

	Lifetime	Thermalisation
Atoms	10s	10ms
Excitons[a]	50ns	0.2ns
Polaritons	5ps	0.5ps
Magnons[b]	1μs(??)	100ns(?)

[a]Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b]Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

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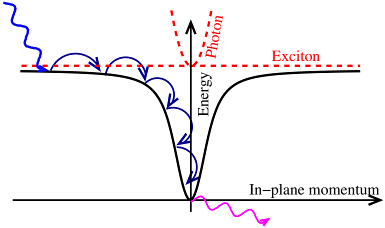
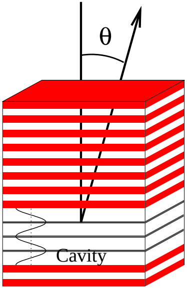
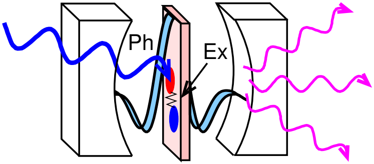


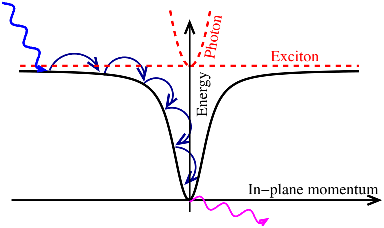


	Lifetime	Thermalisation	Linewidth	Temperature	
Atoms	10s	10ms	2.5×10^{-13}meV	10^{-8}K	10^{-9}meV
Excitons[a]	50ns	0.2ns	5×10^{-5}meV	1K	0.1meV
Polaritons	5ps	0.5ps	0.5meV	20K	2meV
Magnons[b]	1μs(??)	100ns(?)	2.5×10^{-6}meV	300K	30meV

[a] Coupled quantum wells. [Hammack et al PRB 76 193308 (2007)]
[b] Yttrium Iron Garnett. [Demokritov et al, Nature 443 430 (2007)]

Non-equilibrium: flux and baths





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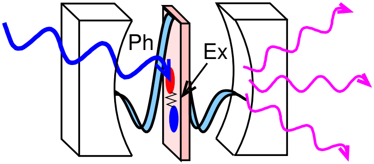
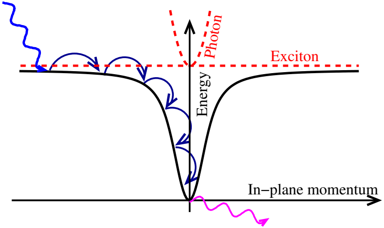
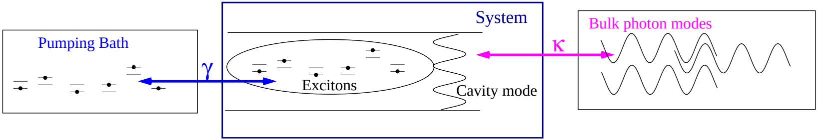
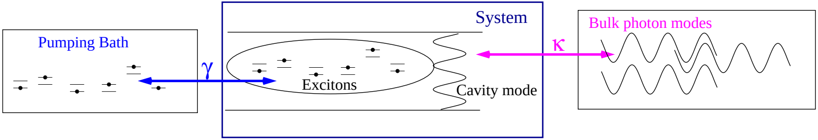






Non-equilibrium model: baths



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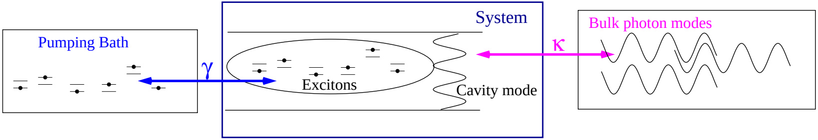

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Schematically: pump γ, decay κ

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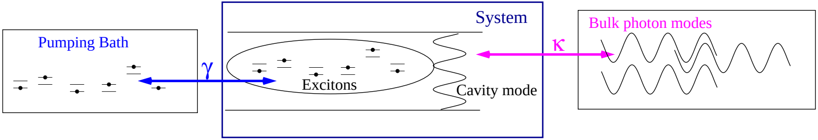

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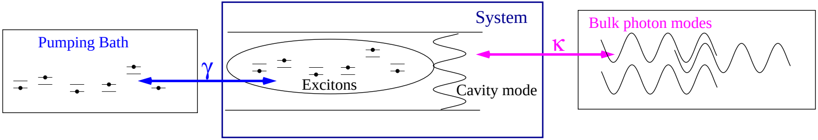

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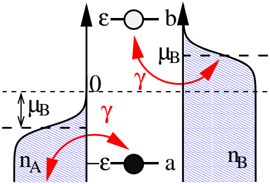

Non-equilibrium theory; mean-field

Look for mean-field solution, $\psi(\mathbf{r}, t) = \psi_0 e^{-i\mu_S t}$. Gap equation:

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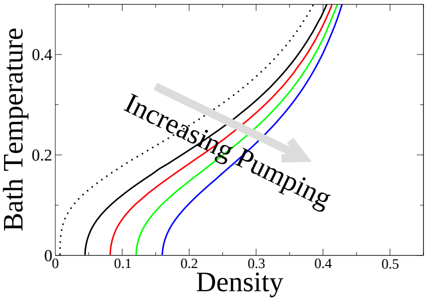

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Gross-Pitaevskii equation: Harmonic trap

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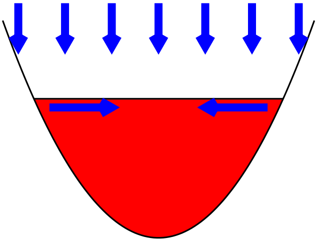

[Keeling & Berloff, PRL, '08]

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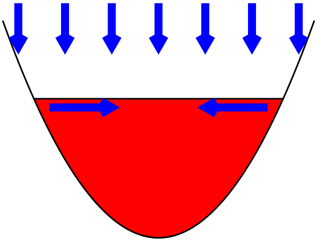
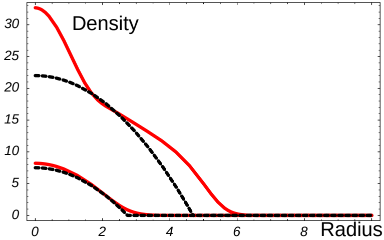
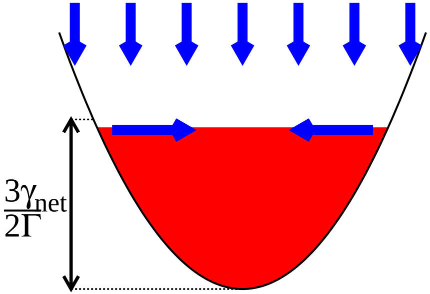




[Keeling & Berloff, PRL, '08]

Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t \rho + \nabla\cdot(\rho \mathbf{v}) = \frac{1}{\hbar}(\gamma_{\text{net}} - \Gamma\rho)\rho$$



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High m modes: $\delta\rho_{n,m} \simeq e^{im\theta} r^m \ldots$

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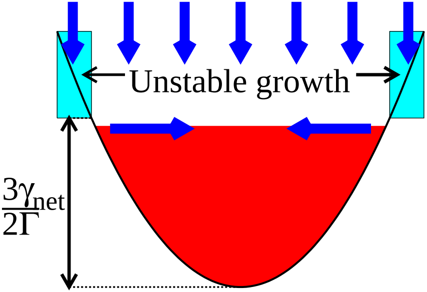

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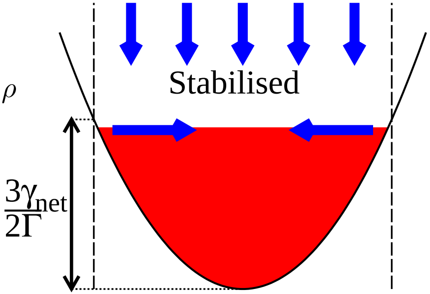

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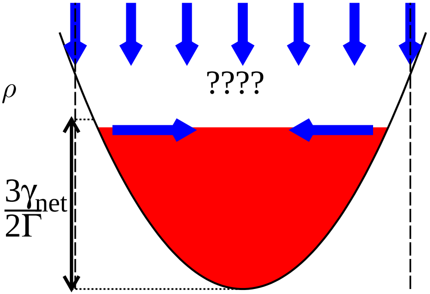
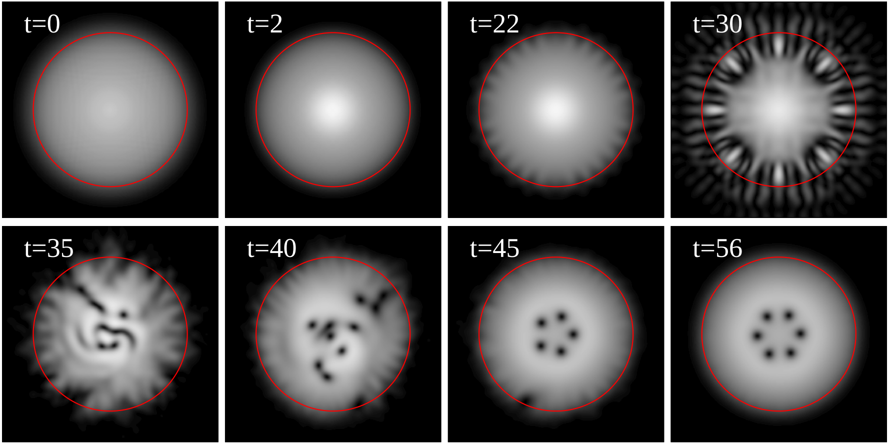


Time evolution:



[Keeling & Berloff, PRL, '08]

Why vortices

Rotating solution: $i\partial_t \psi = (\mu - 2\Omega L_z)\psi$

$$\mathbf{v} = \Omega \times \mathbf{r}, \quad \Omega = \omega, \quad \rho = \frac{\mu}{U} = \frac{\gamma_{\text{net}}}{\Gamma}\Theta(R - r)$$

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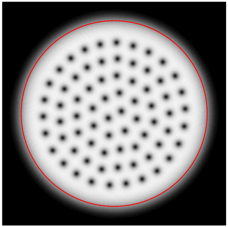

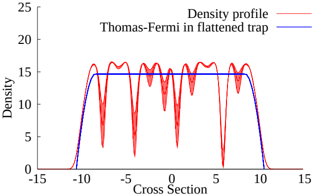

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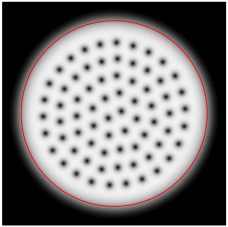
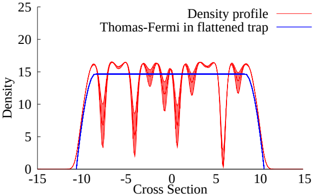
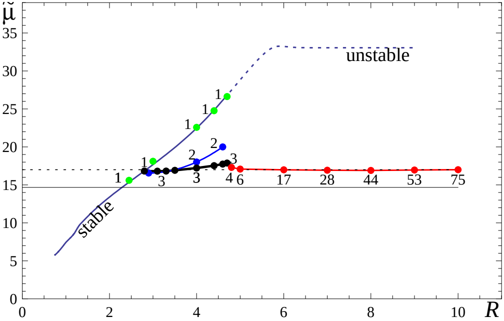






[Keeling & Berloff, PRL, '08]

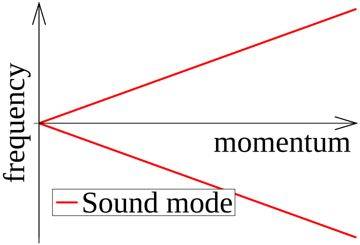
Fluctuations above transition

When condensed

$$\text{Det}\left[\mathcal{G}_R^{-1}(\omega, \mathbf{k})\right] = \omega^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

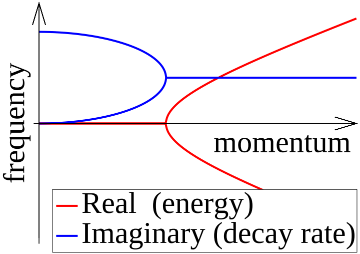
Fluctuations above transition

When condensed

$$\text{Det}\left[\mathcal{G}_R^{-1}(\omega, \mathbf{k})\right] = (\omega + ix)^2 + x^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = -ix \pm \sqrt{c^2\mathbf{k}^2 - x^2}$$



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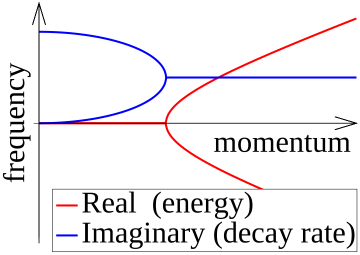

Correlations (in 2D): $\left\langle\psi^\dagger(\mathbf{r},t)\psi(0,r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t,r,r')\right]$

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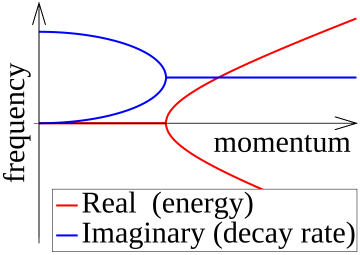

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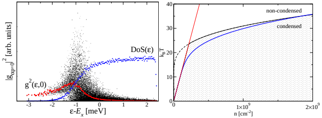
$$\left\langle \psi^\dagger(\mathbf{r}, t)\psi(0, 0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta \begin{cases} \ln(r/\xi) & r \to \infty, t \simeq 0 \\ \frac{1}{2}\ln(c^2 t/x\xi^2) & r \simeq, t \to \infty \end{cases}\right]$$

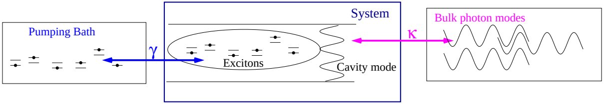
[Szymańska et al., PRL '06; PRB '07]

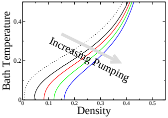
Conclusions

- Localised two-level system model
  Mean-field and fluctuations

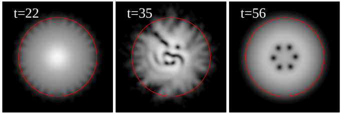


- Effects of pumping on mean-field theory





- Modification to Thomas-Fermi profile
  Spontaneous rotating vortex lattice



Extra slides

6. Equilibrium spectrum
7. Fluctuation corrections
8. Vortices
9. Superfluidity
10. Zero temperature Keldysh boundaries
11. Non-equilibrium Fluctuations
    - Finite size effects: single vs many modes

Equilibrium: Fluctuations about mean-field

Fluctuations $\psi \to \psi + \delta\psi$; correction to action

$$S = S_0 + \sum_{\nu,\mathbf{p},\mathbf{q}} \begin{pmatrix} \delta\psi^*_{\mathbf{p}} \\ \delta\psi_{-\mathbf{p}} \end{pmatrix}^T \mathcal{G}^{-1}_{\mathbf{pq}}(\nu) \begin{pmatrix} \delta\psi_{\mathbf{q}} \\ \delta\psi^*_{-\mathbf{q}} \end{pmatrix}$$

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$$\mathcal{G}^{-1}_{\mathbf{pq}}(\nu) = (\omega_{\mathbf{p}} + i\nu)\mathbb{1}\delta_{\mathbf{pq}} + \sum_{\alpha} g^*_{\alpha\mathbf{p}} g_{\alpha\mathbf{q}} \begin{pmatrix} \chi^{(1)}_{\alpha}(\nu) & \chi^{(2)}_{\alpha}(\nu) \\ \chi^{(2)}_{\alpha}(\nu) & \chi^{(1)*}_{\alpha}(\nu) \end{pmatrix}$$

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- Optical response:
  Treat $\mathbf{p} \neq \mathbf{q}$ perturbatively [D. M. Whittaker PRL 80 4791]
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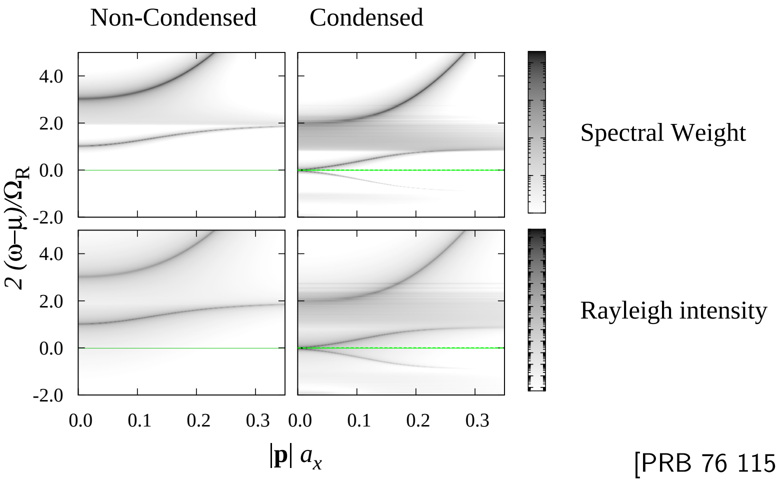
  - Rayleigh scattering $I_{\mathbf{p}\neq\mathbf{q}}(\nu) = |\mathcal{G}^{11}_{\mathbf{pq}}(i\nu)|^2$

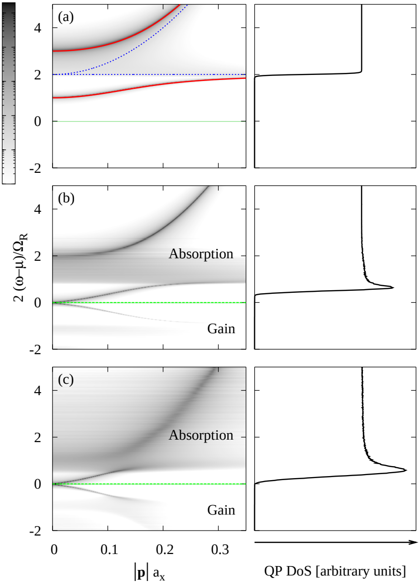
Fluctuations and optical spectra



[PRB 76 115326 (2007)]

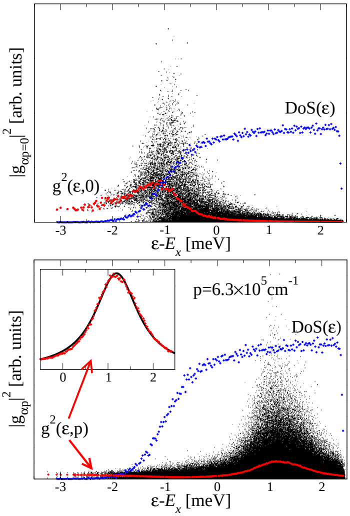
- Phase sensitive Rayleigh → "negative energy" Bogoliubov modes.

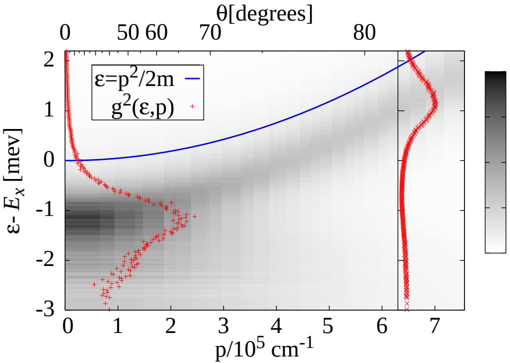
Exciton disorder and polariton density of states



- "Dark" exciton states left at exciton energy.
- Dark states not truly dark, but weak coupling.
- No gap in condensate due to weak coupling tail.

Disorder localised states





Fluctuation corrections to phase boundary

Fluctuation corrections to density

$$\rho \to \rho_0 + \sum_q \langle \delta\psi^\dagger \delta\psi \rangle$$

In 2D system: modified critical condition:

$$\rho_s = \rho - \rho_{\text{normal}} = \# \frac{2m^2}{\hbar} k_B T$$

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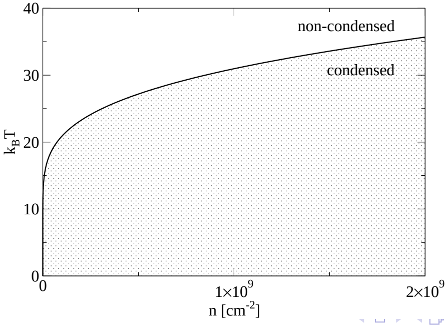

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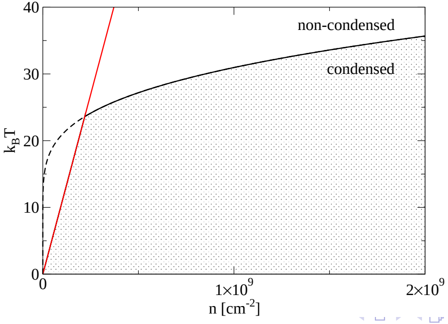

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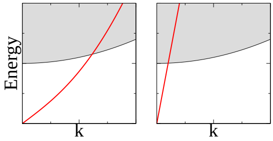

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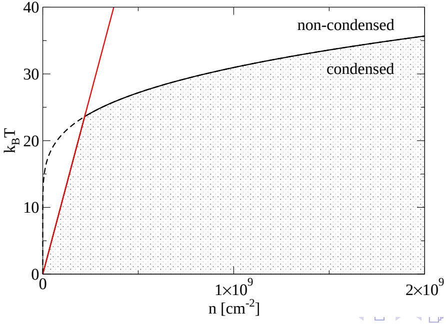

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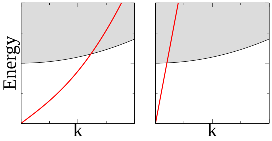

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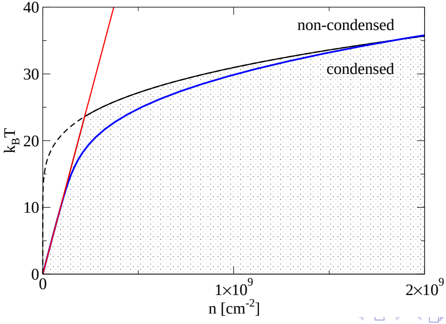

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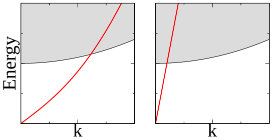

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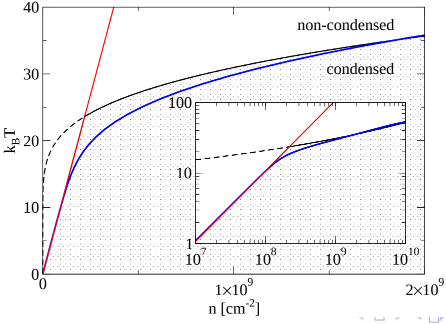
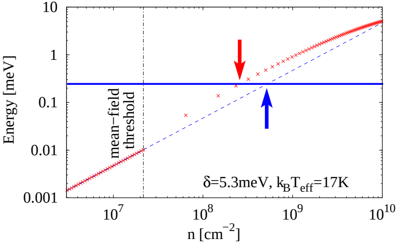


Blueshfit and experimental phase boundary

Blueshift:



Clean limit:

$$\delta E_{\mathrm{LP}} \simeq \mathcal{R}y_{\mathrm{X}} a_{\mathrm{X}}^2 n + \Omega_{\mathrm{R}} a_{\mathrm{X}}^2 n$$

Here: $\Omega_{\mathrm{R}} a_{\mathrm{X}}^2 \rightarrow \Omega_{\mathrm{R}} \xi^2$
[PRB 77 235313]

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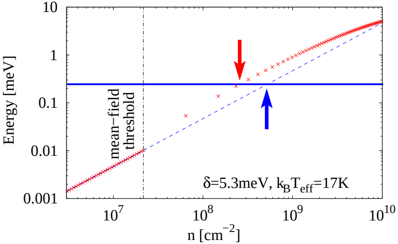

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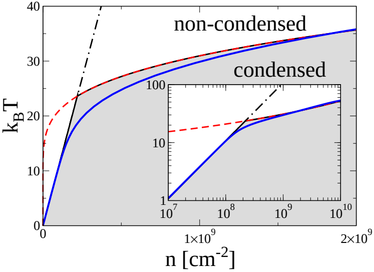
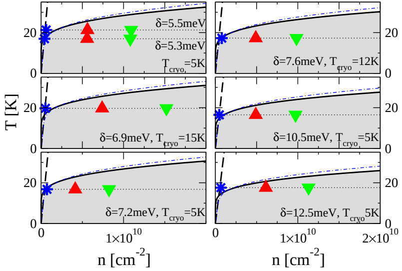
[PRB 77 235313]

Phase diagram:



CdTe:



Instability of Thomas-Fermi: details

$\psi = \sqrt{\rho} e^{i\phi} \quad \mathbf{v} = \frac{\hbar}{m} \nabla \phi$

$$\frac{1}{2}\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = (\gamma_{\text{net}} - \Gamma \rho)\rho$$

$$\partial_t \mathbf{v} + \nabla (U\rho + \frac{m\omega^2}{2} r^2 + \frac{m}{2}|\mathbf{v}|^2) = 0$$

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If $\gamma_{\text{net}}, \Gamma \to 0$, can find normal modes in 2D trap:

$$\delta\rho_{n,m}(r,\theta,t) = e^{im\theta}h_{n,m}(r)e^{i\omega_{n,m}t}$$

$$\omega_{n,m} = \omega 2\sqrt{m(1+2n) + 2n(n+1)}$$

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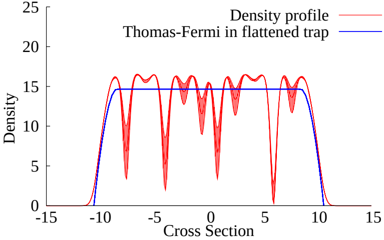
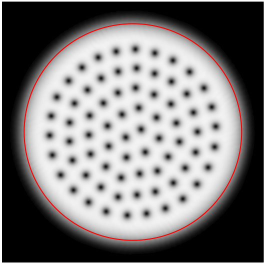
Add weak pumping/decay:

$$\omega_{n,n} \to \omega_{n,m} + i\gamma_{\text{net}} \left[\frac{m(1+2n) + 2n(n+1) - m^2}{2m(1+2n) + 4n(n+1) + m^2} \right]$$

Instability

Why vortices



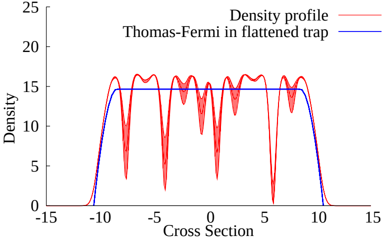


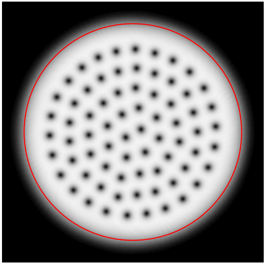
$$\nabla \cdot [\rho(\mathbf{v} - \Omega \times \mathbf{r})] = (\gamma_{net}\Theta(R - r) - \Gamma\rho)\rho$$

$$\mu = \frac{m}{2}|\mathbf{v} - \Omega \times \mathbf{r}|^2 + \frac{m}{2}r^2(\omega^2 - \Omega^2) + U\rho - \frac{\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}$$

$$\mathbf{v} = \Omega \times \mathbf{r}, \quad \Omega = \omega, \quad \rho = \frac{\gamma_{net}}{\Gamma}\Theta(R - r) = \frac{\mu}{U}$$

Why vortices





Rotating solution: $i\partial_t\psi = (\mu - 2\Omega L_z)\psi$

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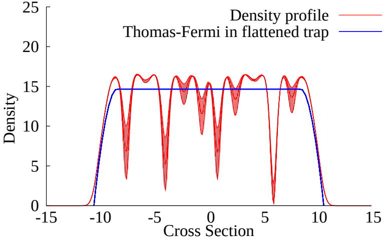

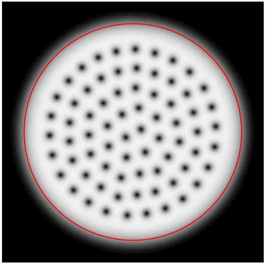

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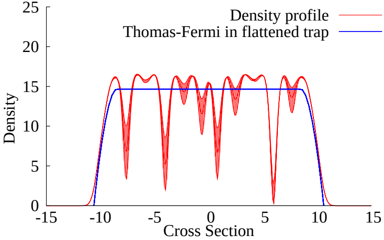

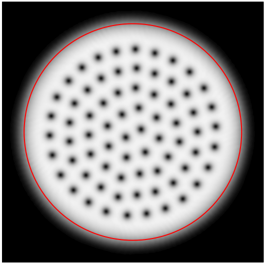

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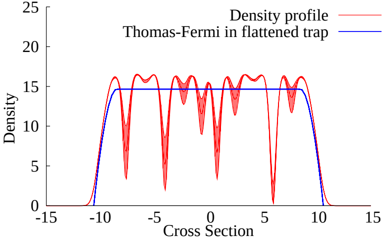

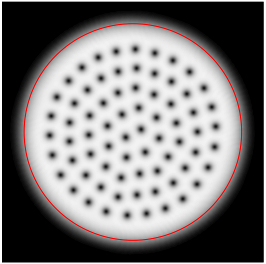

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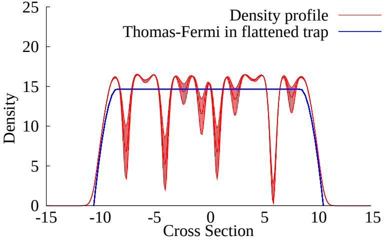

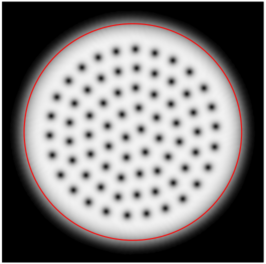

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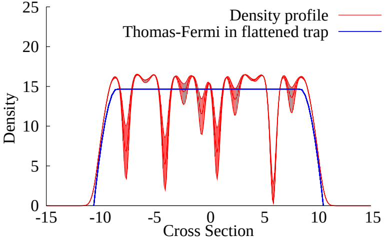

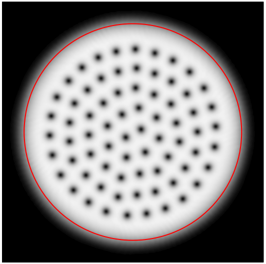

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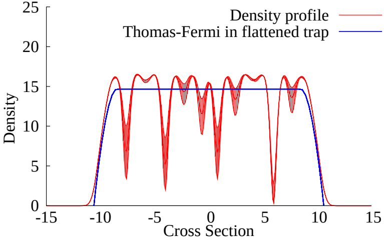

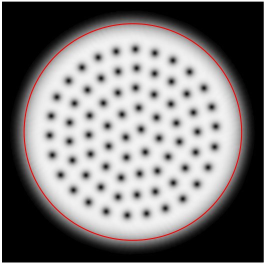

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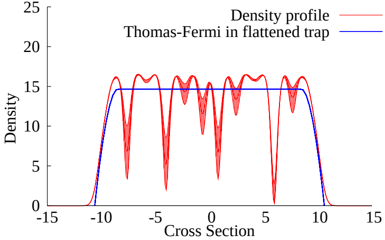

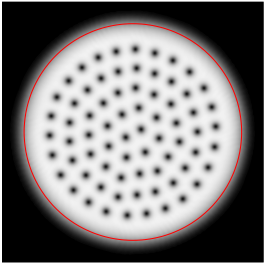

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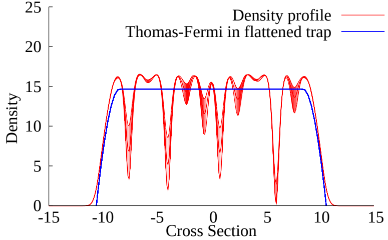

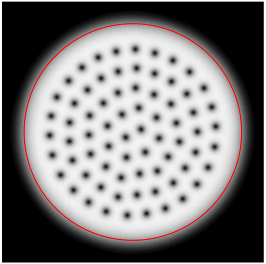

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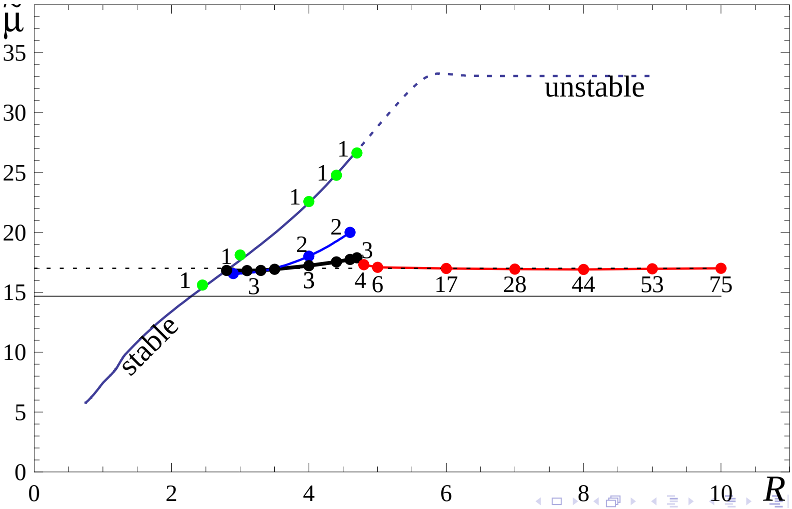
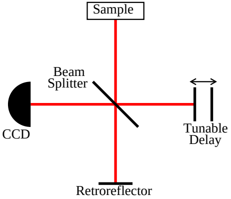
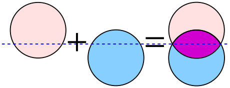
Why vortices: chemical potential vs size

$$\text{Thomas-Fermi}: \mu \propto R^2 \quad \text{Vortex}: \mu = \frac{U\gamma_{\text{net}}}{\Gamma}$$



Observing vortices: fringe pattern





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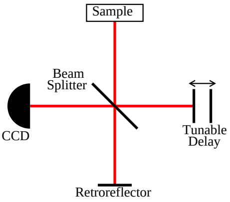

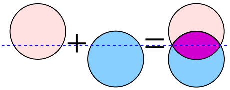

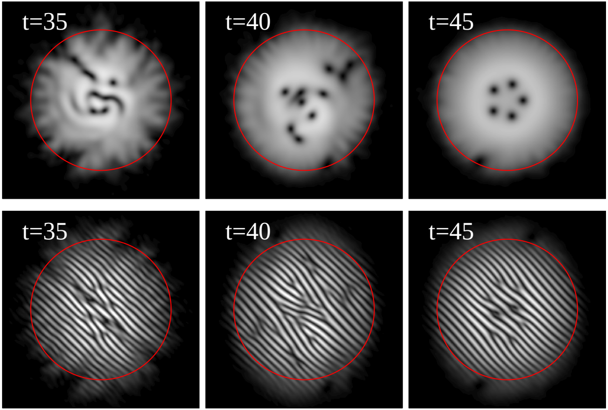

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$$\mathbf{J} = \rho\mathbf{v} = \Psi^\dagger i\hbar\nabla\Psi = |\Psi|^2\nabla\phi$$

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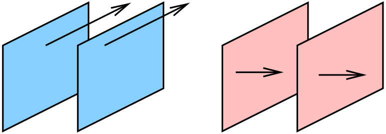

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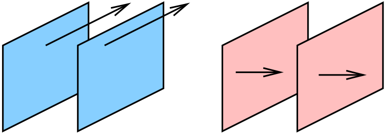

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$\rho_s \propto \lim_{q \to 0} (\chi_L - \chi_T)$.

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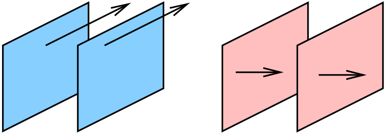

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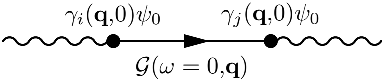
$\Delta\chi_{ij}(q) =$  $+ \dots$

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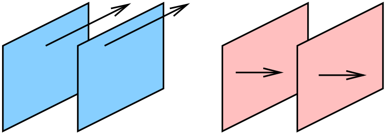

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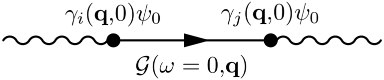
$\Delta\chi_{ij}(q) =$  $+ \ldots$

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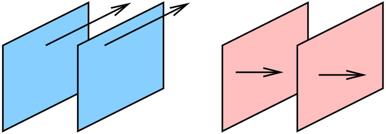

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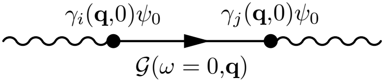
$\Delta\chi_{ij}(q) =$ 

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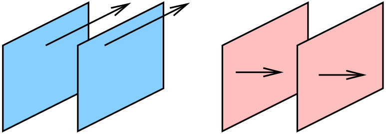

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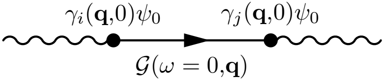
Static ρ_S survies

$\Delta\chi_{ij}(q) =$  $+ \dots$

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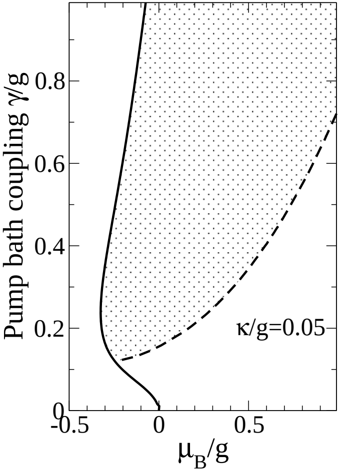
Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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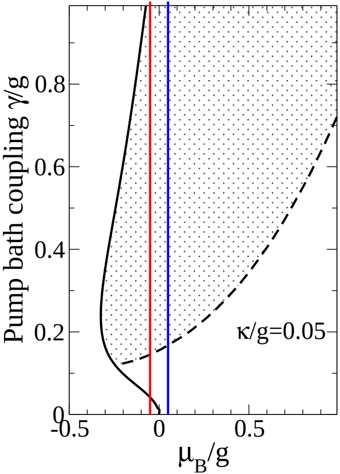

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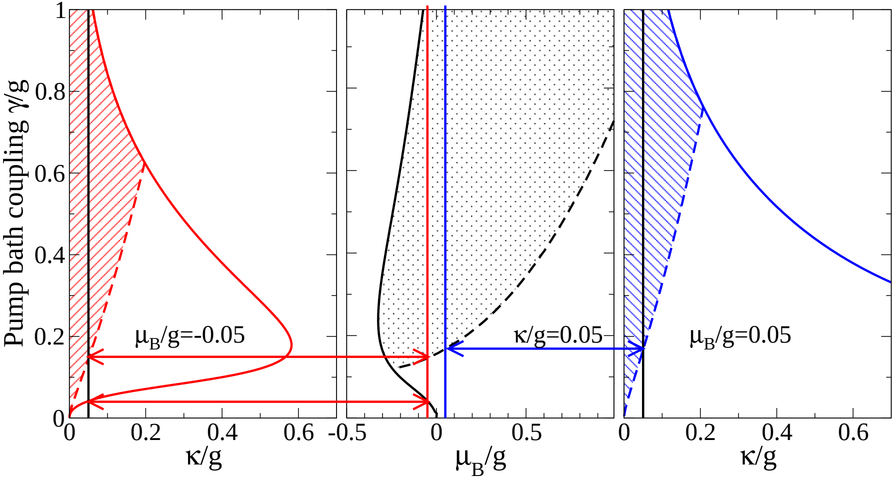

Fluctuations → Stability, Luminescence, Absorption

Keldysh approach:

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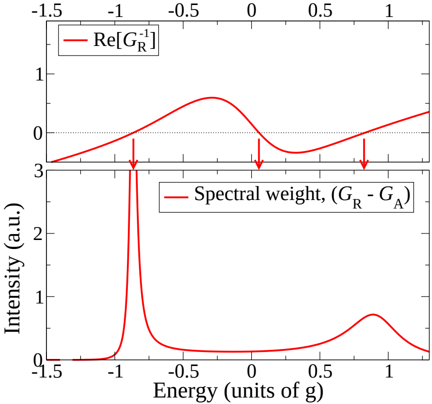

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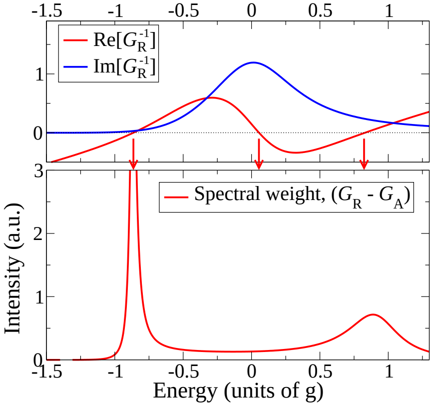

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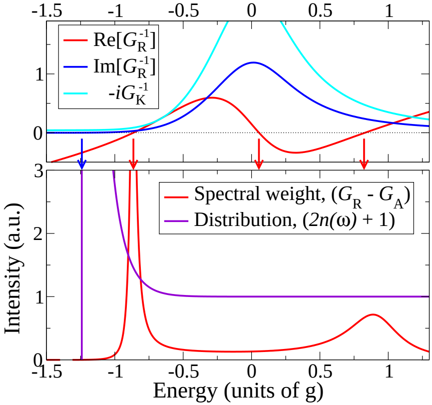
$$\mathcal{G}_R^{-1}(\omega, k) = (\omega - \omega_k^*) + i\alpha(\omega - \mu_{\text{eff}})$$



Fluctuations → Stability, Luminescence, Absorption

Keeldysh approach:

$$\mathcal{G}_R - \mathcal{G}_A = -i\left\langle \left[\psi^\dagger, \psi\right]_- \right\rangle$$

$$\mathcal{G}_K = -i\left\langle \left[\psi^\dagger, \psi\right]_+ \right\rangle = (2n(\omega)+1)(\mathcal{G}_R - \mathcal{G}_A)$$

$$\mathcal{L} = \left\langle \psi^\dagger \psi \right\rangle = \frac{i}{2}\left[\mathcal{G}_K + (\mathcal{G}_R - \mathcal{G}_A)\right]$$

$$\mathcal{G}_K = \frac{-{\mathcal{G}_\mathcal{K}}^{-1}}{\mathrm{Im}[{\mathcal{G}_\mathcal{R}}^{-1}]^2 + \mathrm{Re}[{\mathcal{G}_\mathcal{R}}^{-1}]^2}$$

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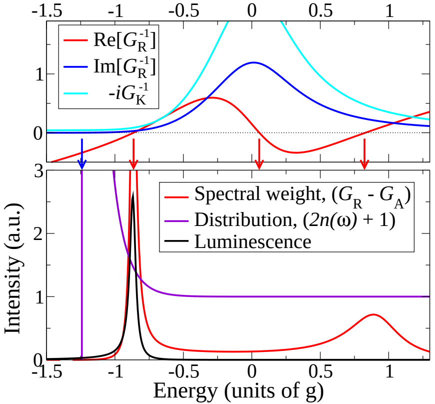
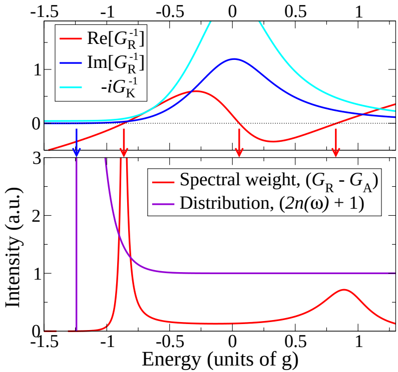


Linewidth, inverse Green's function and gap equation



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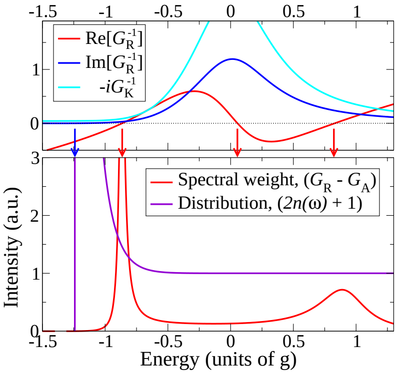

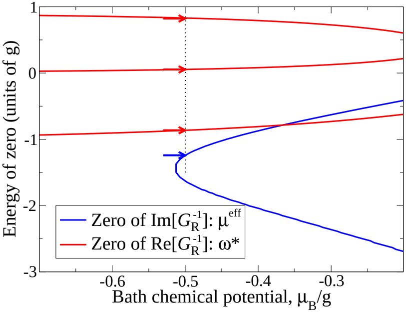

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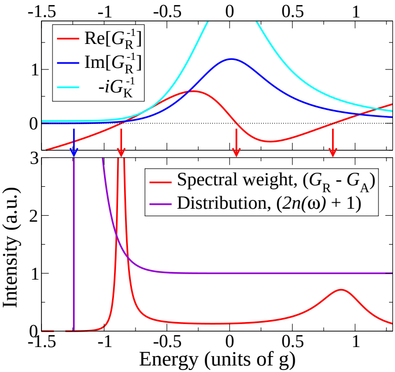
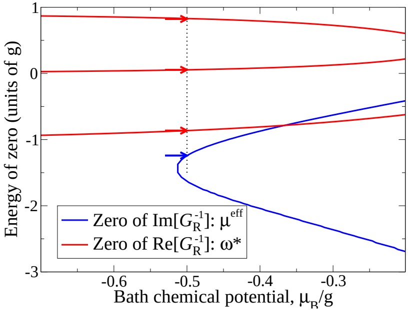
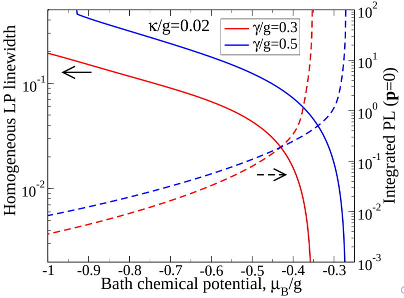






At transition, Gap Equation implies:

$$\mathcal{G}_R^{-1}(\omega = \mu_S, k = 0) = 0$$

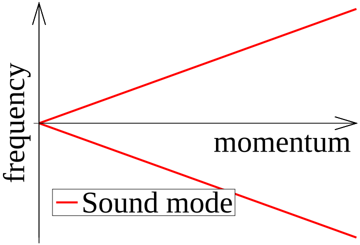
Fluctuations above transition

When condensed

Det $\left[\mathcal{G}_R^{-1}(\omega,\mathbf{k})\right] = \omega^2 - c^2\mathbf{k}^2$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

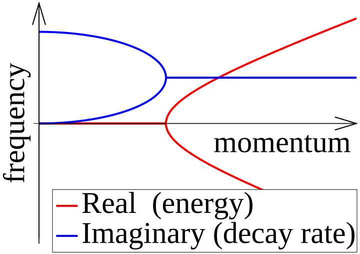
Fluctuations above transition

When condensed

$$\text{Det}\left[\mathcal{G}_R^{-1}(\omega, \mathbf{k})\right] = (\omega + ix)^2 + x^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = -ix \pm \sqrt{c^2\mathbf{k}^2 - x^2}$$



[Szymańska et al., PRL '06; PRB '07]

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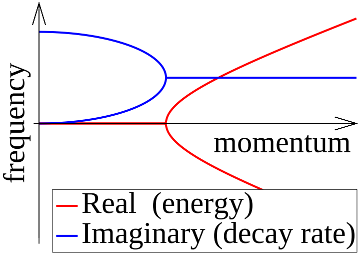

Correlations (in 2D): $\left\langle \psi^\dagger(\mathbf{r}, t)\psi(0, r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t, r, r')\right]$

[Szymańska et al., PRL '06; PRB '07]

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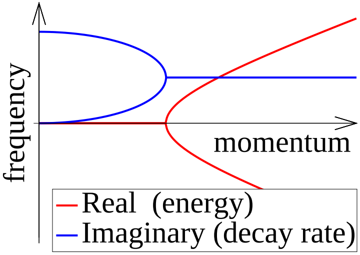

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$$\left\langle \psi^\dagger(\mathbf{r},t)\psi(0,0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta\begin{cases}\ln(r/\xi) & r\to\infty, t\simeq 0\\ \frac{1}{2}\ln(c^2t/x\xi^2) & r\simeq, t\to\infty\end{cases}\right]$$

[Szymańska et al., PRL '06; PRB '07]

Finite size effects: Single mode vs many mode

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$\mathcal{D}_{\phi\phi}(t,r,r')$ from sum of phase modes. Study $ct \gg r$ limit:

$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_{n}^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{\left|(\omega+ix)^2+x^2-(n\Delta)^2\right|^2}$$

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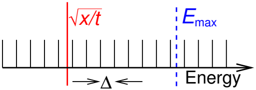
$\Delta \ll \sqrt{x/t} \ll E_{\text{max}}$



$\mathcal{D}_{\phi\phi} \sim 1 + \ln(E_{\text{max}}\sqrt{t/x})$

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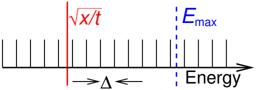

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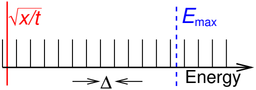

$\mathcal{D}_{\phi\phi} \sim \left(\frac{\pi C}{2x}\right)\left(\frac{t}{2x}\right)$

Relating finite-size spectrum to self phase modulation

Finite-size spectrum:

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Connection to Kubo Formula [Eastham and Whittaker, arXiv:0811.4333]

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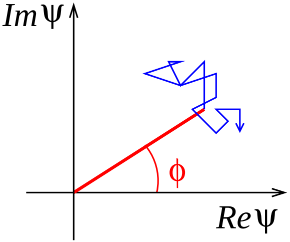

$$\partial_t \phi = U \delta N$$

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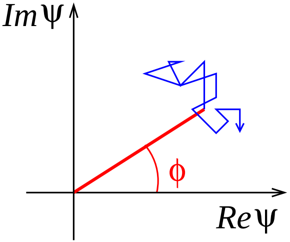

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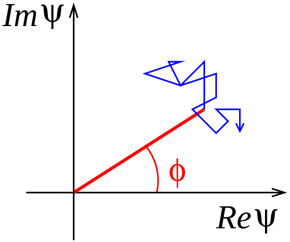

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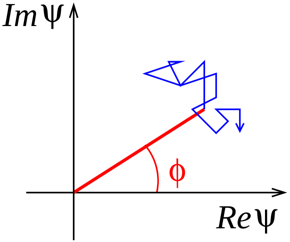

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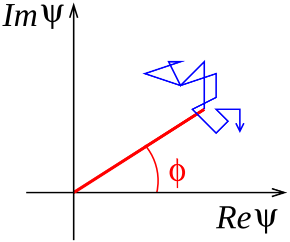

$$\begin{aligned}
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Relating finite-size spectrum to self phase modulation

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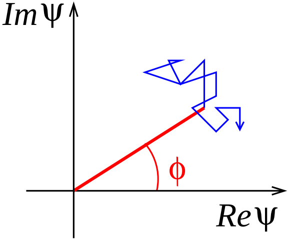

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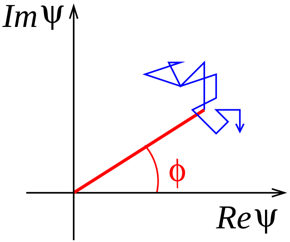

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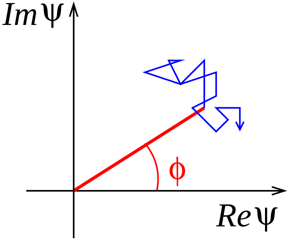

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