Nonequilibrium quantum condensates: from microscopic theory to macroscopic phenomenology

N. G. Berloff, **J. M. J. Keeling**, P. B. Littlewood, F. M. Marchetti, M. H. Szymanska.

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Overview

1. Model and mean-field theory
   - Disorder-localised exciton model
   - Connections of mean-field equation to other limits
2. Macroscopic phenomenology
   - Gross Pitaevskii equation in an harmonic trap
   - Spontaneously rotating vortex lattice
3. Fluctuations and correlations
   - Fluctuations about mean-field theory
   - Finite size effects: single vs many modes

Polariton system model

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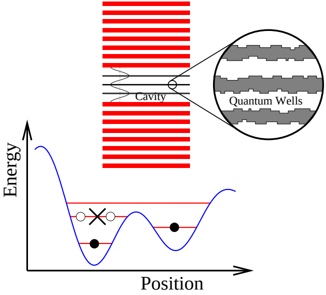
- Disorder-localised excitons
- Treat disorder sites as two-level (exciton/no-exciton)
- Propagating (2D) photons
- Exciton–photon coupling g.



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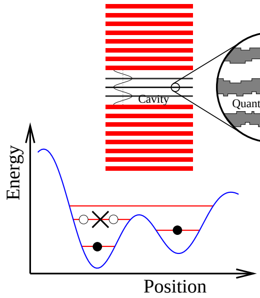
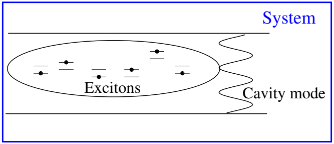
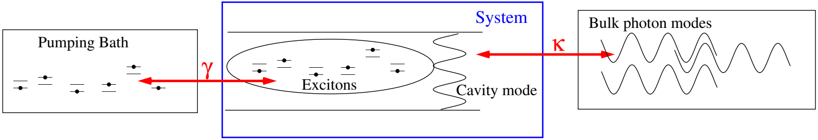


$$H_{sys} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \psi^\dagger_{\mathbf{k}} \psi_{\mathbf{k}} + \sum_{\alpha} \left[\epsilon_\alpha \left(b^\dagger_\alpha b_\alpha - a^\dagger_\alpha a_\alpha \right) + \frac{1}{\sqrt{\text{Area}}} g_{\alpha,\mathbf{k}} \psi_{\mathbf{k}} b^\dagger_\alpha a_\alpha + \text{H.c.} \right]$$



Non-equilibrium model: baths



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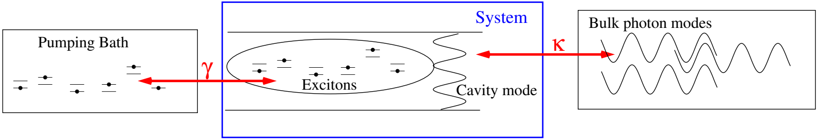

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Schematically: pump γ, decay κ

$$H_{\text{sys,bath}} \simeq \sum_{\mathbf{p},\vec{k}} \sqrt{\kappa}\psi_{\mathbf{k}}\Psi^{\dagger}_{\mathbf{p}} + \sum_{\alpha,\beta} \sqrt{\gamma}\left(a^{\dagger}_{\alpha}A_{\beta} + b^{\dagger}_{\alpha}B_{\beta}\right) + \text{H.c.}$$

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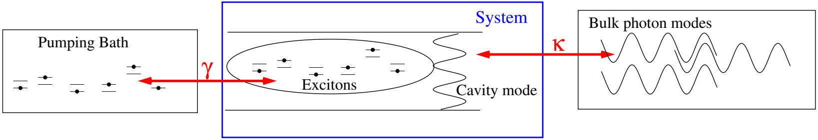

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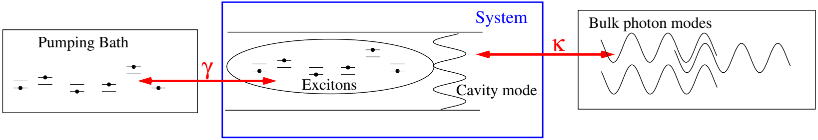

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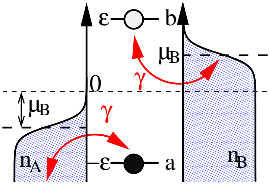
Ψ bath is empty. Pumping bath thermal, μ_B, T:



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Susceptibility:

$$\chi(\psi_0, \mu_s) = -g^2\gamma \sum_{\text{excitons}} \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(i\gamma + \epsilon_\alpha - \frac{1}{2}\mu_S)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}$$

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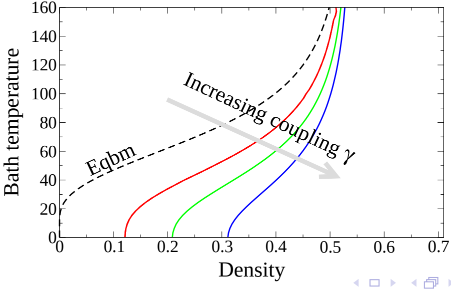

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$$\omega_0 - \mu_s = \frac{g^2}{2E} \tanh\left(\frac{\beta E}{2}\right)$$

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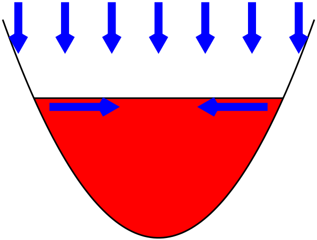

[Keeling & Berloff, PRL, '08]

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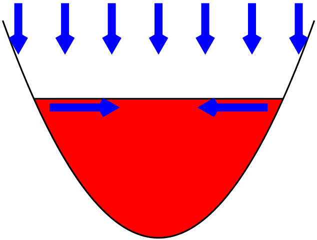
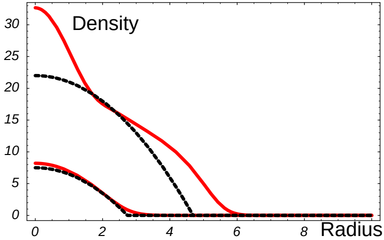
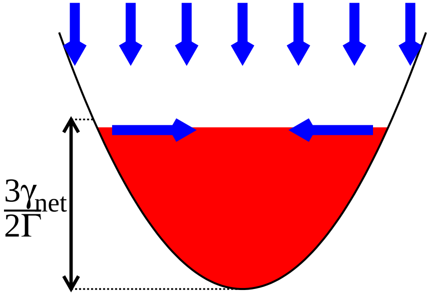




[Keeling & Berloff, PRL, '08]

Stability of Thomas-Fermi solution

$$\frac{1}{2}\partial_t\rho+\nabla\cdot(\rho\mathbf{v}) = \frac{1}{\hbar}(\gamma_{\mathrm{net}} - \Gamma\rho)\rho$$



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High m modes: $\delta\rho_{n,m} \simeq e^{im\theta} r^m \ldots$

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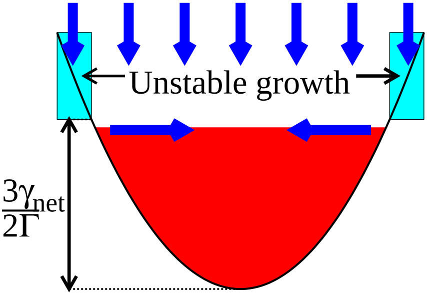

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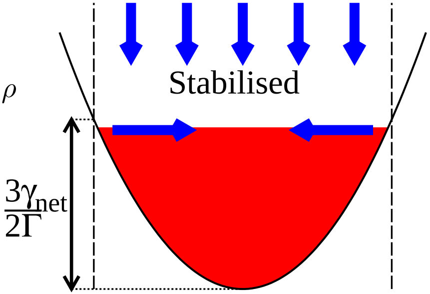

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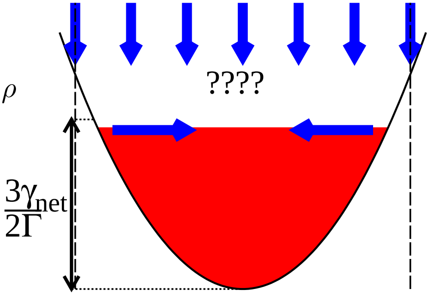
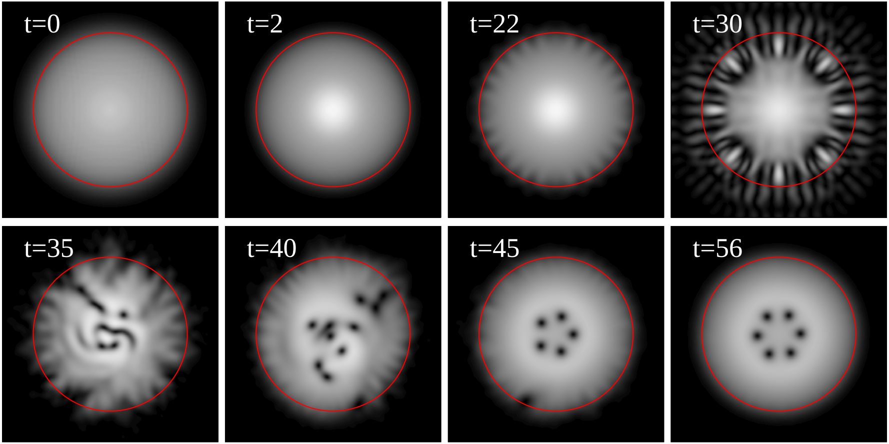


Time evolution:



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Fluctuations → Stability, Luminescence, Absorption

Keldysh approach:

$$\mathcal{G}_{R,A} = \mp i\theta[\pm(t-t')]\left\langle \left[\psi^\dagger, \psi\right]_- \right\rangle$$

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$$\begin{aligned} \mathcal{G}_K &= -\frac{1}{\mathcal{G_R}^{-1}} \mathcal{G}_K^{-1} \frac{1}{[\mathcal{G_R}^{-1}]^\dagger} \\ \mathcal{G}_R - \mathcal{G}_A &= \frac{1}{\mathcal{G_R}^{-1}} - \frac{1}{[\mathcal{G_R}^{-1}]^\dagger} \end{aligned}$$

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$$\mathcal{G}_R - \mathcal{G}_A = \frac{2\mathrm{Im}[{\mathcal{G}_{\mathcal{R}}}^{-1}]}{\mathrm{Im}[{\mathcal{G}_{\mathcal{R}}}^{-1}]^2 + \mathrm{Re}[{\mathcal{G}_{\mathcal{R}}}^{-1}]^2}$$

Fluctuations → Stability, Luminescence, Absorption

Keldysh approach:

$$\mathcal{G}_R - \mathcal{G}_A = -i\left\langle\left[\psi^\dagger, \psi\right]_-\right\rangle$$

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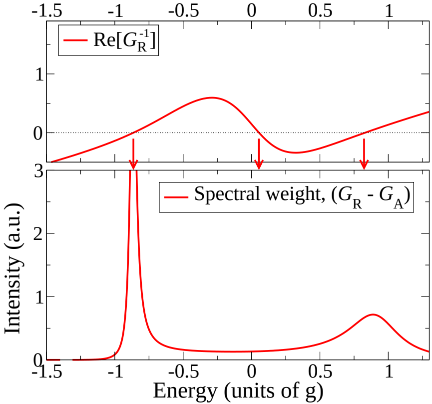
$$\mathcal{G}_R^{-1}(\omega, k) = (\omega - \omega_k^*)$$



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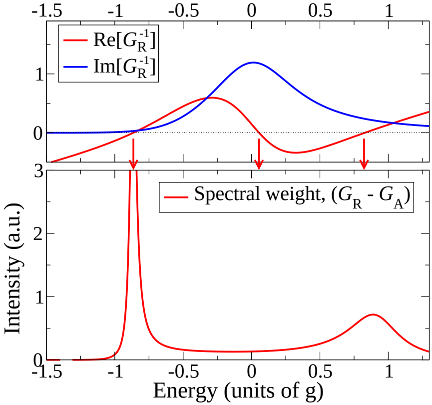

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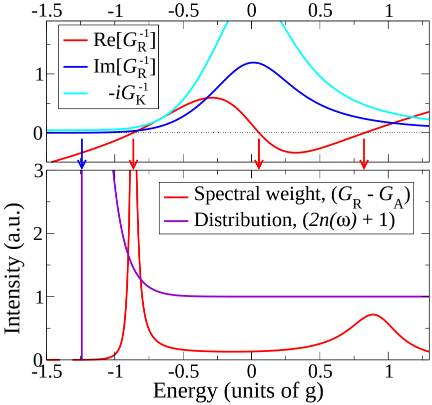

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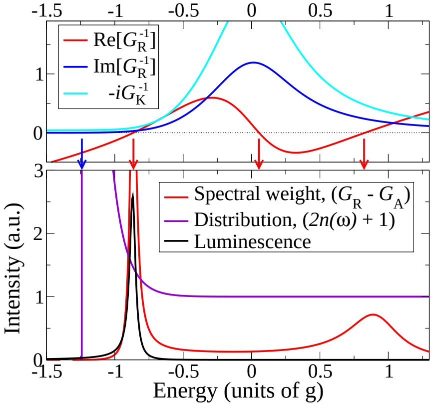
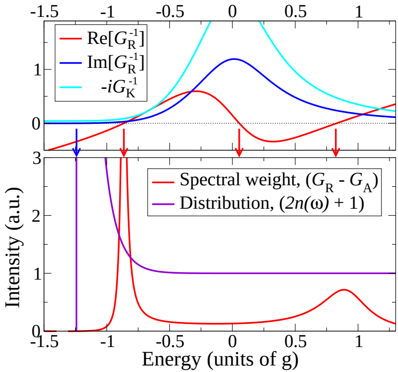


Linewidth, inverse Green's function and gap equation



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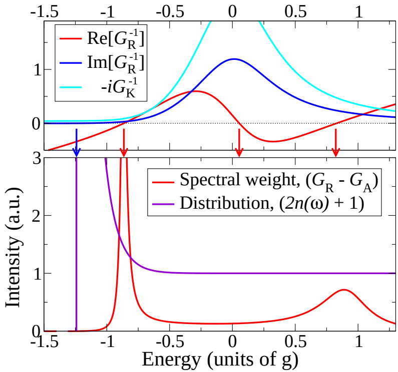

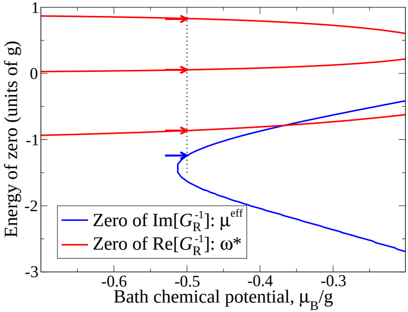

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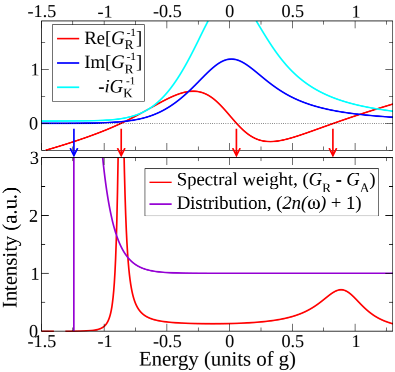
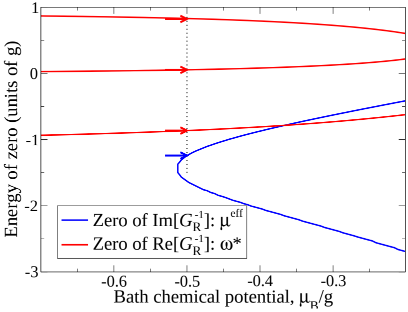
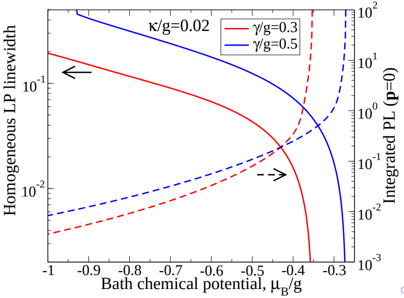






At transition, Gap Equation implies:

$$\mathcal{G}_R^{-1}(\omega = \mu_S, k = 0) = 0$$

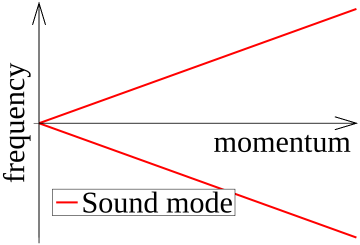
Fluctuations above transition

When condensed

$$\text{Det}\left[\mathcal{G}_R^{-1}(\omega, k)\right] = \omega^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = c|\mathbf{k}|$$



[Szymańska et al., PRL '06; PRB '07]

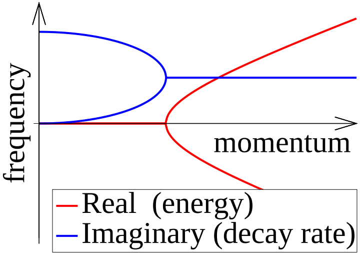
Fluctuations above transition

When condensed

$$\text{Det}\left[\mathcal{G}_R^{-1}(\omega, k)\right] = (\omega + ix)^2 + x^2 - c^2\mathbf{k}^2$$

Poles:

$$\omega^* = -ix \pm \sqrt{c^2\mathbf{k}^2 - x^2}$$



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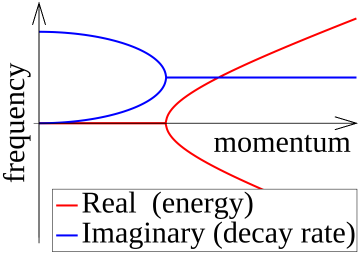

Correlations (in 2D): $\left\langle \psi^\dagger(\mathbf{r}, t)\psi(0, r')\right\rangle \simeq |\psi_0|^2 \exp\left[-\mathcal{D}_{\phi\phi}(t, r, r')\right]$

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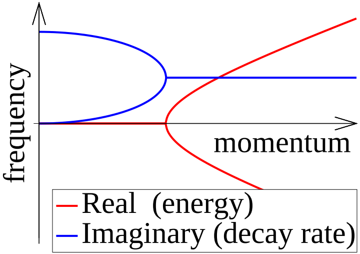

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$$\left\langle\psi^\dagger(\mathbf{r}, t)\psi(0, 0)\right\rangle \simeq |\psi_0|^2 \exp\left[-\eta\begin{cases}\ln(r/\xi) & r \to \infty, t \simeq 0\\ \frac{1}{2}\ln(c^2 t/x\xi^2) & r \simeq, t \to \infty\end{cases}\right]$$

[Szymańska et al., PRL '06; PRB '07]

Finite size effects: Single mode vs many mode

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$\mathcal{D}_{\phi\phi}(t,r,r')$ from sum of phase modes. Study $ct \gg r$ limit:

$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_{n}^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{|(\omega+ix)^2 + x^2 - (n\Delta)^2|^2}$$

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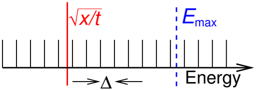
$\Delta \ll \sqrt{x/t} \ll E_{\max}$



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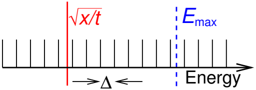

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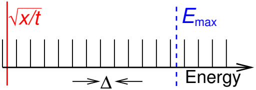
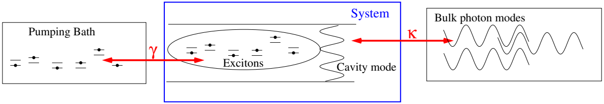
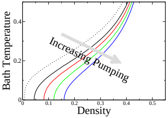


$\mathcal{D}_{\phi\phi} \sim \left(\frac{\pi C}{2x}\right)\left(\frac{t}{2x}\right)$

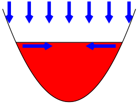
Conclusions

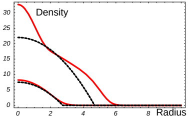
- Effects of pumping on mean-field theory



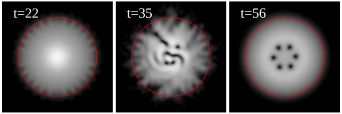


- Modification to Thomas-Fermi profile

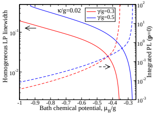


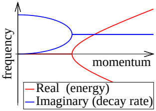


- Spontaneous rotating vortex lattice



- Change to spectrum and correlations

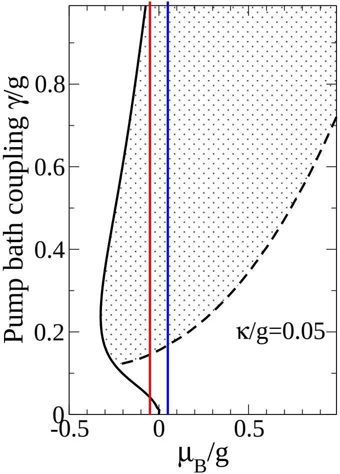




Extra slides

Zero temperature phase diagram

$$\tilde{\omega}_0 - i\kappa = g^2\gamma \sum_\alpha \int \frac{d\nu}{2\pi} \frac{(F_a + F_b)\nu + (F_b - F_a)(\tilde{\epsilon}_\alpha + i\gamma)}{[(\nu - E_\alpha)^2 + \gamma^2][(\nu + E_\alpha)^2 + \gamma^2]}.$$



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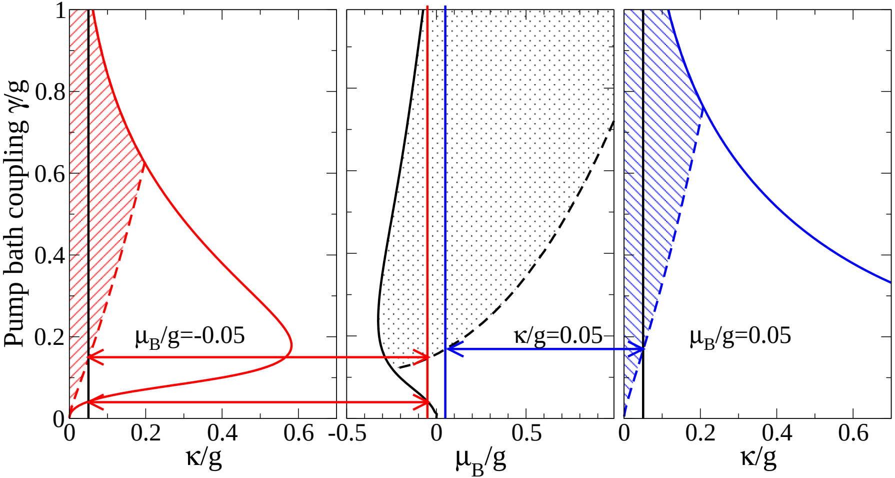
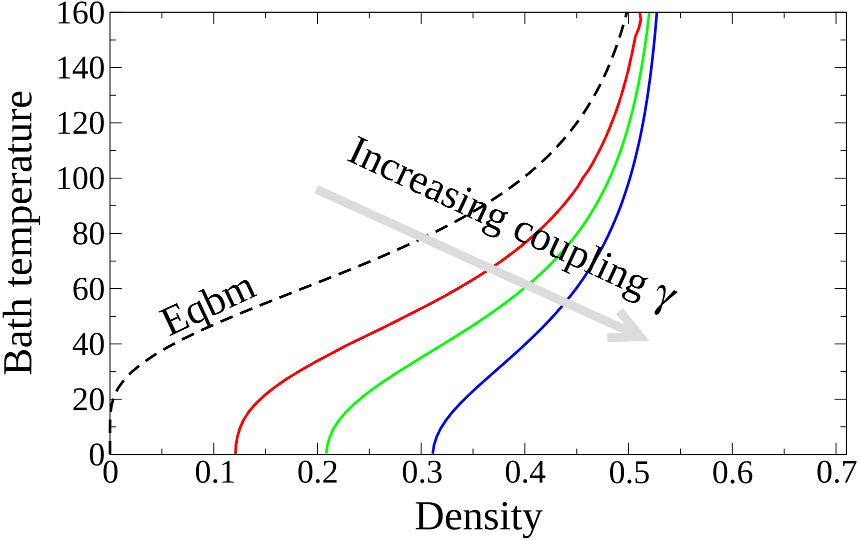


Condensate fraction



Instability of Thomas-Fermi: details

$\psi = \sqrt{\rho}e^{i\phi} \quad \mathbf{v} = \nabla\phi$

$$\frac{1}{2}\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = (\alpha - \sigma\rho)\rho$$
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If $\alpha, \sigma \to 0$, can find normal modes in 2D trap:

$$\delta\rho_{n,m}(r, \theta, t) = e^{im\theta} h_{n,m}(r) e^{i\omega_{n,m} t}$$
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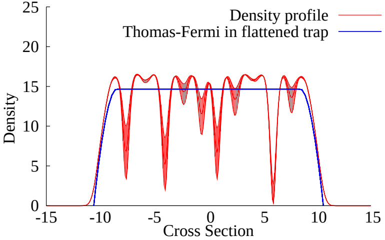
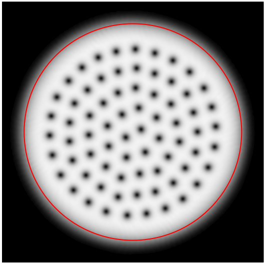
Add weak pumping/decay:

$$\omega_{n,n} \to \omega_{n,m} + i\alpha\left[\frac{m(1+2n) + 2n(n+1) - m^2}{2m(1+2n) + 4n(n+1) + m^2}\right]$$

Instability

Why vortices





$$\nabla \cdot [\rho(\mathbf{v} - \Omega \times \mathbf{r})] = (\alpha\Theta(R - r) - \sigma\rho)\,\rho,$$

$$\mu = |\mathbf{v} - \Omega \times \mathbf{r}|^2 + r^2(1 - \Omega^2) + \rho - \frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}$$

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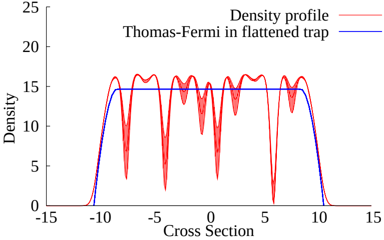

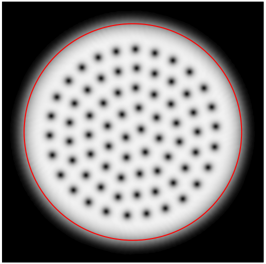

Rotating solution: $i\partial_t \psi = (\mu - 2\Omega L_z)\psi$

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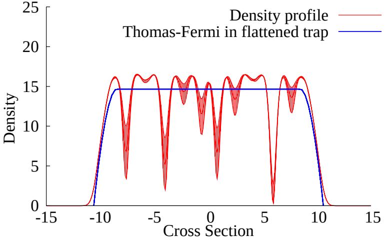

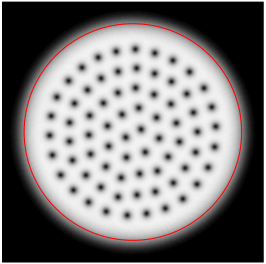

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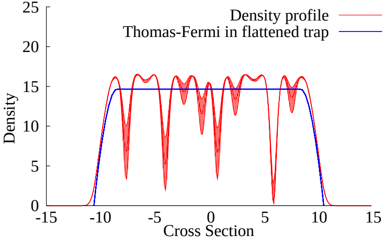

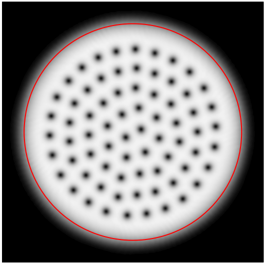

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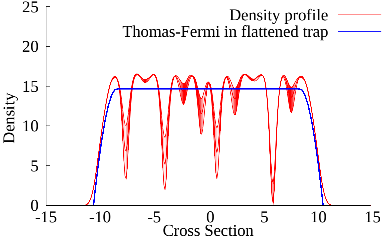

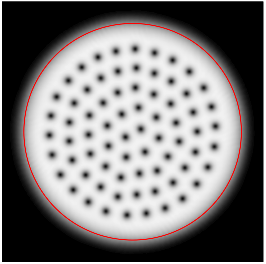

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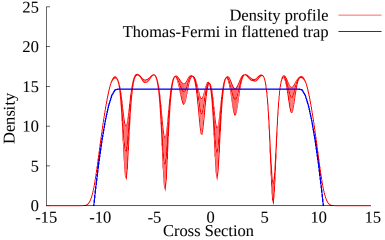

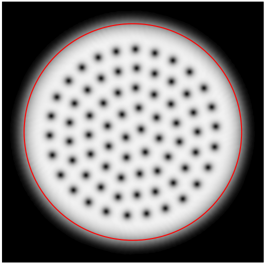

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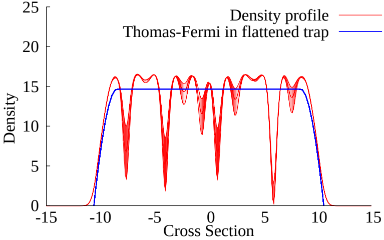

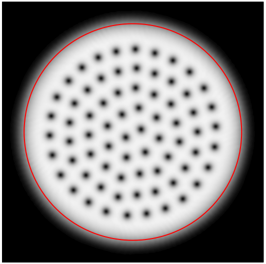

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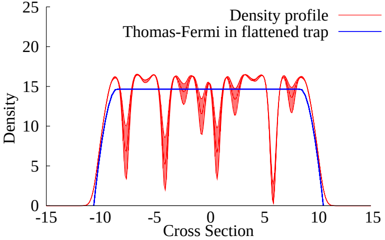

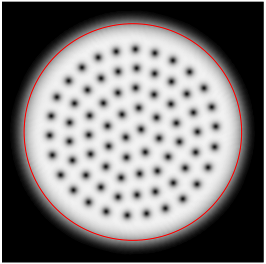

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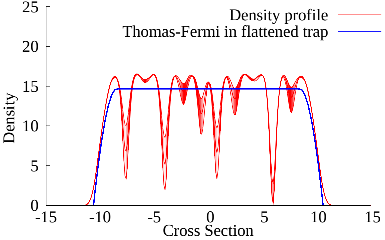

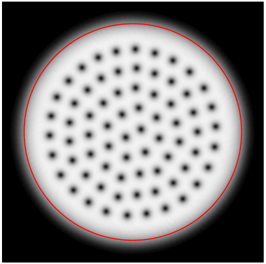

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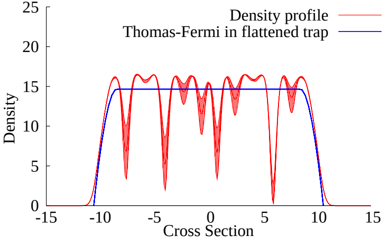

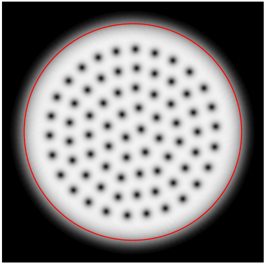

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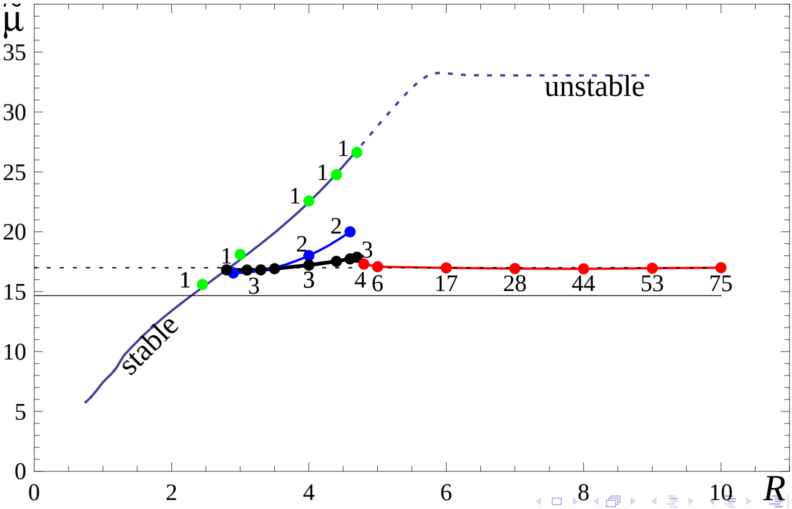
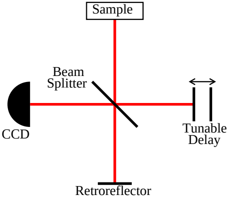
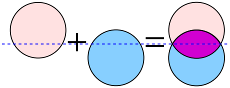
Why vortices: chemical potential vs size

$$\text{Thomas-Fermi} : \mu = R^2 \quad \text{Vortex} : \mu = \frac{\alpha}{\sigma}$$



Observing vortices: fringe pattern





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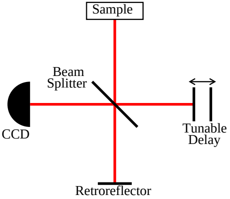

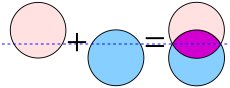

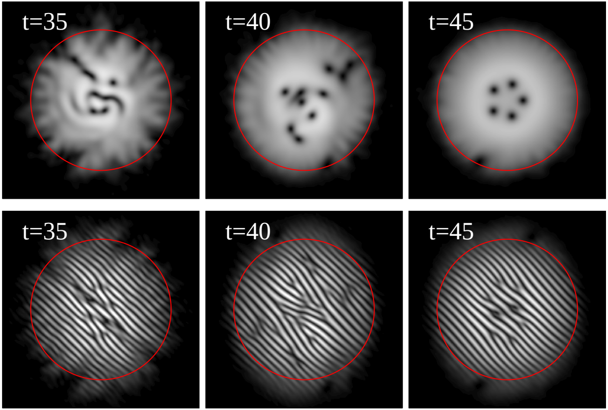

Relating finite-size spectrum to self phase modulation

$$\mathcal{D}_{\phi\phi}(t,r,r) \propto \sum_{n}^{n_{max}} \int \frac{d\omega}{2\pi} \frac{|\varphi_n(r)|^2(1-e^{i\omega t})}{\left|(\omega+ix)^2+x^2-(n\Delta)^2\right|^2}$$

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$$\partial_t \phi = U\delta N$$

Basic idea of SPM:

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Basic idea of SPM:

$$\begin{aligned}\partial_t \phi &= U\delta N \\ \partial_t N &= -\Gamma N + F(t), \quad \langle F(t)F(t')\rangle = C\delta(t-t') \\ \mathcal{D}_{\phi\phi}(t) &= \int \frac{d\omega}{2\pi}(1-e^{i\omega t})\frac{CU^2}{|\omega(\omega+i\Gamma)|^2} \\ &= \frac{CU^2}{2\Gamma^2}\left[t - \frac{(1-e^{-\Gamma t})}{\Gamma}\right]\end{aligned}$$

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Thus: $2x \simeq \Gamma$