10. Integrals of basic functions

Try to find all the integrals in this section without referring to a table of integrals. The integrals of these functions occur so frequently that you should try to memorise the appropriate rules. If you are really stuck, consult the Tables on page 4.

1. Integrate each of the following with respect to x.

(a) x^4 (b) x^7 (c) $x^{1/2}$ (d) $x^{1/3}$ (e) $\sqrt{x}$ (f) $x^{-1/2}$ (g) $\sqrt[4]{x}$

(h) $\dfrac{1}{x^3}$ (i) $x^{0.2}$ (j) $\dfrac{1}{x^{0.3}}$ (k) $\dfrac{1}{\sqrt{x}}$ (l) $\dfrac{1}{\sqrt{x^3}}$ (m) x^{-2} (n) $x^{4/3}$

2. Integrate each of the following with respect to x.

(a) $\cos 5x$ (b) $\sin 2x$ (c) $\sin \frac{1}{2}x$ (d) $\cos \dfrac{x}{2}$ (e) $\dfrac{1}{x}$ (f) e^{2x}

(g) e^{-2x} (h) $\mathrm{e}^{x/3}$ (i) $\mathrm{e}^{0.5x}$ (j) $\dfrac{1}{\mathrm{e}^x}$ (k) $\dfrac{1}{\mathrm{e}^{2x}}$ (l) $\cos(-7x)$

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11. Linearity in integration

The **linearity rules** enable us to integrate sums (and differences) of functions, and constant multiples of functions. Specifically

$$\int (f(x) \pm g(x))\mathrm{d}x = \int f(x)\,\mathrm{d}x \pm \int g(x)\mathrm{d}x, \qquad \int k\,f(x)\mathrm{d}x = k\int f(x)\mathrm{d}x$$

1. Integrate each of the following with respect to x.

(a) $7x^4$ (b) $-4x^7$ (c) $x^{1/2} + x^{1/3}$ (d) $17x^{1/3}$ (e) $\sqrt{x} - \dfrac{1}{\sqrt{x}}$ (f) $x^2 + \dfrac{1}{x}$

(g) $x^3 + \dfrac{1}{x^2}$ (h) $\dfrac{1}{7x^3}$ (i) 11 (j) $\dfrac{11}{x^{0.3}}$ (k) $2x - \dfrac{2}{x}$ (l) $7x - 11$

2. Integrate each of the following with respect to x.

(a) $3x + \cos 4x$ (b) $4 + \sin 3x$ (c) $\dfrac{x}{2} + \sin\dfrac{x}{2}$ (d) $4\mathrm{e}^x + \cos\dfrac{x}{2}$

(e) $\mathrm{e}^{-2x} + \mathrm{e}^{2x}$ (f) $3\sin 2x + 2\sin 3x$ (g) $\dfrac{1}{kx}$, k constant (h) $-1 - \dfrac{4}{x}$

(i) $1 + x + x^2$ (j) $\dfrac{1}{3x} - 7$ (k) $\frac{1}{2}\cos\frac{1}{2}x$ (l) $\frac{1}{2}x^2 - 3x^{-1/2}$

3. Simplify each of the following expressions first and then integrate them with respect to x.

(a) $6x(x+1)$ (b) $(x+1)(x-2)$ (c) $\dfrac{x^3 + 2x^2}{\sqrt{x}}$ (d) $(\sqrt{x}+2)(\sqrt{x}-3)$

(e) $e^{2x}(e^x - e^{-x})$ (f) $\dfrac{e^{3x} - e^{2x}}{e^x}$ (g) $\dfrac{x+4}{x}$ (h) $\dfrac{x^2 + 3x + 2}{x+2}$

12. Evaluating definite integrals

1. Evaluate each of the following definite integrals.

(a) $\int_0^1 7x^4 \mathrm{d}x$ (b) $\int_{-2}^3 -4t^7 \mathrm{d}t$ (c) $\int_1^2 (x^{1/2}+x^{1/3})\mathrm{d}x$ (d) $\int_{-2}^{-1} 17t^{1/3}\mathrm{d}t$

(e) $\int_1^3 (2s+8s^3)\mathrm{d}s$ (f) $\int_1^5 \frac{1}{x^2}\mathrm{d}x$ (g) $\int_0^3 (t^2+2t)\mathrm{d}t$ (h) $\int_0^{\pi/4} \cos 2x \, \mathrm{d}x$

(i) $\int_0^{1/2} \mathrm{e}^{3x}\mathrm{d}x$ (j) $\int_2^4 \frac{1}{\sqrt{\mathrm{e}^x}}\mathrm{d}x$ (k) $\int_0^{\pi/4} (2\lambda + \sin\lambda)\mathrm{d}\lambda$ (l) $\int_0^1 (\mathrm{e}^x + \mathrm{e}^{-x})\mathrm{d}x$

2. For the function $f(x) = x^2 + 3x - 2$ verify that

$$\int_0^2 f(x)\mathrm{d}x + \int_2^3 f(x)\mathrm{d}x = \int_0^3 f(x)\mathrm{d}x$$

3. For the function $f(x) = 4x^2 - 7x$ verify that $\int_{-1}^1 f(x)\mathrm{d}x = -\int_1^{-1} f(x)\mathrm{d}x$.

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13. Integration by parts

Integration by parts is a technique which can often be used to integrate products of functions. If u and v are both functions of x then

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

When dealing with definite integrals the relevant formula is

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

1. Integrate each of the following with respect to x.

(a) $x\mathrm{e}^x$ (b) $5x\cos x$ (c) $x\sin x$ (d) $x\cos 2x$ (e) $x\ln x$ (f) $2x\sin\frac{x}{2}$

2. Evaluate the following definite integrals.

(a) $\int_0^{\pi/2} x\cos x \, \mathrm{d}x$ (b) $\int_1^2 4x\mathrm{e}^x\mathrm{d}x$ (c) $\int_{-1}^1 5t\mathrm{e}^{-2t}\mathrm{d}t$ (d) $\int_1^3 x\ln x \, \mathrm{d}x$

3. In the following exercises it may be necessary to apply the integration by parts formula more than once. Integrate each of the following with respect to x.

(a) $x^2\mathrm{e}^x$ (b) $5x^2\cos x$ (c) $x(\ln x)^2$ (d) $x^3\mathrm{e}^{-x}$ (e) $x^2\sin\frac{x}{2}$

4. Evaluate the following definite integrals.

(a) $\int_0^{\pi/2} x^2\cos x \, \mathrm{d}x$ (b) $\int_0^1 7x^2\mathrm{e}^x\mathrm{d}x$ (c) $\int_{-1}^1 t^2\mathrm{e}^{-2t}\mathrm{d}t$

5. By writing $\ln x$ as $1 \times \ln x$ find $\int \ln x \, \mathrm{d}x$.

6. Let I stand for the integral $\int \frac{\ln t}{t}\mathrm{d}t$. Using integration by parts show that

$$I = (\ln t)^2 - I \qquad \text{(plus a constant of integration)}$$

Hence deduce that $I = \frac{1}{2}(\ln t)^2 + c$.

7. Let I stand for the integral $\int e^t \sin t \, dt$. Using integration by parts twice show that

$$I = e^t \sin t - e^t \cos t - I \qquad \text{(plus a constant of integration)}$$

Hence deduce that $\int e^t \sin t \, dt = \dfrac{e^t(\sin t - \cos t)}{2} + c$

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14. Integration by substitution

1. Find each of the following integrals using the given substitution.

(a) $\int \cos(x-3)dx, \quad u = x-3$ (b) $\int \sin(2x+4)dx, \quad u = 2x+4$

(c) $\int \cos(\omega t + \phi)dt, \quad u = \omega t + \phi$ (d) $\int e^{9x-7}dx, \quad u = 9x-7$

(e) $\int x(3x^2+8)^3 dx, \quad u = 3x^2+8$ (f) $\int x\sqrt{4x-3}\,dx, \quad u = 4x-3$

(g) $\int \dfrac{1}{(x-2)^4}dx, \quad u = x-2$ (h) $\int \dfrac{1}{(3-t)^5}dt, \quad u = 3-t$

(i) $\int x\,e^{-x^2}dx, \quad u = -x^2$ (j) $\int \sin x \, \cos^3 x \, dx, \quad u = \cos x$

(k) $\int \dfrac{t}{\sqrt{1+t^2}}dt, \quad u = 1+t^2$ (l) $\int \dfrac{x}{2x+1}dx, \quad u = 2x+1$

2. Find each of the following integrals using the given substitution.

(a) $\int \dfrac{x}{\sqrt{x-3}}dx, \quad z = \sqrt{x-3}$ (b) $\int (x-5)^4(x+3)^2dx, \quad u = x-5$

3. Evaluate each of the following definite integrals using the given substitution.

(a) $\int_0^{\pi/4} \cos(x-\pi)dx, \quad z = x-\pi$ (b) $\int_8^9 (x-8)^5(x+1)^2dx, \quad u = x-8$

(c) $\int_2^3 t\sqrt{t-2}\,dt,$ by letting $u = t-2$, and also by letting $u = \sqrt{t-2}$

(d) $\int_1^2 t(t^2+5)^3dt, \quad u = t^2+5$ (e) $\int_0^{\pi/4} \tan x \sec^2 x \, dx, \quad u = \tan x$

(f) $\int_{\pi^2/4}^{\pi^2} \dfrac{\sin\sqrt{x}}{\sqrt{x}}dx, \quad u = \sqrt{x}$ (g) $\int_1^2 \dfrac{e^{\sqrt{t}}}{\sqrt{t}}dt, \quad u = \sqrt{t}$

4. By means of the substitution $u = 4x^2 - 7x + 2$ show that

$$\int \frac{8x-7}{4x^2-7x+2}dx = \ln|4x^2-7x+2| + c$$

5. The result of the previous exercise is a particular case of a more general rule with which you should become familiar: when the integrand takes the form

$$\frac{\text{derivative of denominator}}{\text{denominator}}$$

the integral is the logarithm of the denominator. Use this rule to find the following integrals, checking each example by making an appropriate substitution.

(a) $\int \dfrac{1}{x+1}dx$ (b) $\int \dfrac{1}{x-3}dx$ (c) $\int \dfrac{3}{3x+4}dx$ (d) $\int \dfrac{2}{2x+1}dx$

6. Use the technique of Question 5 together with a linearity rule to find the following integrals. For example, to find $\int \frac{x}{x^2-7}\,\mathrm{d}x$ we note that the numerator can be made equal to the derivative of the denominator as follows:

$$\int \frac{x}{x^2-7}\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2-7}\mathrm{d}x = \frac{1}{2}\ln|x^2-7| + c.$$

(a) $\int \frac{x}{x^2+1}\mathrm{d}x$ (b) $\int \frac{\sin 3\theta}{1+\cos 3\theta}\mathrm{d}\theta$ (c) $\int \frac{3e^{2x}}{1+e^{2x}}\,\mathrm{d}x$

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15. Integration using partial fractions

1. By expressing the integrand as the sum of its partial fractions, find the following integrals.

(a) $\int \frac{2x+1}{x^2+x}\,\mathrm{d}x$ (b) $\int \frac{3x+1}{x^2+x}\,\mathrm{d}x$ (c) $\int \frac{5x+6}{x^2+3x+2}\,\mathrm{d}x$

(d) $\int \frac{x+1}{(1-x)(x-2)}\,\mathrm{d}x$ (e) $\int \frac{9x+25}{x^2+10x+9}\,\mathrm{d}x$ (f) $\int \frac{5x-11}{x^2+10x+9}\,\mathrm{d}x$

(g) $\int \frac{4x}{4-x^2}\,\mathrm{d}x$ (h) $\int \frac{15x+51}{x^2+7x+10}\,\mathrm{d}x$ (i) $\int \frac{\mathrm{d}s}{s^2-1}$

2. By expressing the integrand as the sum of its partial fractions, find the following definite integrals.

(a) $\int_1^2 \frac{2-8x}{x^2+2x}\,\mathrm{d}x$ (b) $\int_0^2 \frac{5x+7}{(x+1)(x+2)}\,\mathrm{d}x$ (c) $\int_{-1}^0 \frac{7x-11}{x^2-3x+2}\,\mathrm{d}x$

3. By expressing the integrand as the sum of its partial fractions, find the following integrals.

(a) $\int \frac{x}{x^2-2x+1}\,\mathrm{d}x$ (b) $\int \frac{4x+6}{(x+1)^2}\,\mathrm{d}x$ (c) $\int \frac{7x-23}{x^2-6x+9}\,\mathrm{d}x$

4. By expressing the integrand as the sum of its partial fractions, find the following definite integrals.

(a) $\int_0^1 \frac{x+8}{x^2+6x+9}\,\mathrm{d}x$ (b) $\int_{-1}^1 \frac{2x+19}{x^2+18x+81}\,\mathrm{d}x$ (c) $\int_0^2 -\frac{8x}{x^2+2x+1}\,\mathrm{d}x$

5. Find $\int \frac{2x^2+6x+5}{(x+2)(x+1)^2}\mathrm{d}x$.

6. Find $\int \frac{5x^2+x-34}{(x-2)(x-3)(x+4)}\,\mathrm{d}x$.

7. Find $\int \frac{x^2-11x-4}{(2x+1)(x+1)(3-x)}\,\mathrm{d}x$.

8. In this example note that the degree of the numerator is greater than the degree of the denominator. Find $\int \frac{x^3}{(x+1)(x+2)}\,\mathrm{d}x$.

9. Show that $\dfrac{2x^3+1}{(x+1)(x+2)^2}$ can be written in the form

$$2 - \frac{1}{x+1} + \frac{15}{(x+2)^2} - \frac{9}{x+2}$$

Hence find $\displaystyle\int \frac{2x^3+1}{(x+1)(x+2)^2}\,dx$.

10. Find $\displaystyle\int \frac{4x^3+10x+4}{2x^2+x}\,dx$ by expressing the integrand in partial fractions.

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16. Integration using trigonometrical identities

Trigonometrical identities can often be used to write an integrand in an alternative form which can then be integrated. Some identities which are particularly useful for integration are given in the table below.

Table of trigonometric identities

$$\sin A\,\sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$
$$\cos A\,\cos B = \tfrac{1}{2}\left(\cos(A-B) + \cos(A+B)\right)$$
$$\sin A\,\cos B = \tfrac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$
$$\sin A\,\cos A = \tfrac{1}{2}\sin 2A$$
$$\cos^2 A = \tfrac{1}{2}(1+\cos 2A)$$
$$\sin^2 A = \tfrac{1}{2}(1-\cos 2A)$$
$$\sin^2 A + \cos^2 A = 1$$
$$\tan^2 A = \sec^2 A - 1$$

1. In preparation for what follows find each of the following integrals.

(a) $\int \sin 3x\,dx$ (b) $\int \cos 8x\,dx$ (c) $\int \sin 7t\,dt$ (d) $\int \cos 6x\,dx$

2. Use the identity $\sin^2 A = \frac{1}{2}(1-\cos 2A)$ to find $\int \sin^2 x\,dx$.

3. Use the identity $\sin A\,\cos A = \frac{1}{2}\sin 2A$ to find $\int \sin t\cos t\,dt$.

4. Find (a) $\int 2\sin 7t\cos 3t\,dt$ (b) $\int 8\cos 9x\cos 4x\,dx$ (c) $\int \sin t\sin 7t\,dt$

5. Find $\int \tan^2 t\,dt$.
(Hint: use one of the identities and note that the derivative of $\tan t$ is $\sec^2 t$).

6. Find $\int \cos^4 t\,dt$. (Hint: square an identity for $\cos^2 A$).

7. (a) Use the substitution $u = \cos x$ to show that

$$\int \sin x \cos^n x \, dx = -\frac{1}{n+1} \cos^{n+1} x + c$$

(b) In this question you are required to find the integral $\int \sin^5 t \, dt$. Start by writing the integrand as $\sin^4 t \sin t$. Take the identity $\sin^2 t = 1 - \cos^2 t$ and square it to produce an identity for $\sin^4 t$. Finally use the result in part (a) to find the required integral, $\int \sin^5 t \, dt$.

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17. Miscellaneous integration exercises

To find the integrals in this section you will need to select an appropriate technique from any of the earlier techniques.

1. Find $\int (9x-2)^5 dx$.

2. Find $\int \frac{1}{\sqrt{t}+1} dt$.

3. Find $\int t^4 \ln t \, dt$.

4. Find $\int (5\sqrt{t} - 3t^3 + 2) \, dt$.

5. Find $\int (\cos 3t + 3 \sin t) \, dt$.

6. By taking logarithms to base e show that a^x can be written as $e^{x \ln a}$. Hence find $\int a^x \, dx$ where a is a constant.

7. Find $\int x e^{3x+1} \, dx$.

8. Find $\int \frac{2x}{(x+2)(x-2)} \, dx$.

9. Find $\int \tan\theta \sec^2\theta \, d\theta$.

10. Find $\int_0^{\pi/2} \sin^3 x \, dx$.

11. Using the substitution $x = \sin\theta$ find $\int \sqrt{1-x^2} \, dx$.

12. Find $\int e^x \sin 2x \, dx$.

13. Find $\displaystyle\int \frac{x-9}{x(x-1)(x+3)}\,\mathrm{d}x$.

14. Find $\displaystyle\int \frac{x^3+2x^2-10x-9}{(x-3)(x+3)}\,\mathrm{d}x$.

15. Let I_n stand for the integral $\displaystyle\int x^n \mathrm{e}^{2x}\mathrm{d}x$. Using integration by parts show that $I_n = \dfrac{x^n \mathrm{e}^{2x}}{2} - \dfrac{n}{2} I_{n-1}$. This result is known as a **reduction formula**. Use it repeatedly to find I_4, that is $\displaystyle\int x^4 \mathrm{e}^{2x}\mathrm{d}x$.

16. Find $\displaystyle\int_0^1 \frac{1}{(4-t^2)^{3/2}}\mathrm{d}t$, by letting $t = 2\sin\theta$.