4. The product rule for differentiation

The rule for differentiating the product of two functions $f(x)$ and $g(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x).$$

1. Differentiate each of the following with respect to x.

(a) $x \sin x$ (b) $x^3 \cos 2x$ (c) $x^{-1/3}\, e^{-3x}$

(d) $\sqrt{x}\ln 4x$ (e) $(x^2 - x)\sin 6x$ (f) $\dfrac{1}{x}\left(\tan\dfrac{x}{3} - \cos\dfrac{x}{3}\right)$

2. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta\,\cos\theta)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\sin 2t\,\tan 5t)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}z}(\sin z\,\ln 4z)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x/2}\cos\dfrac{x}{2}\right)$

(e) $\dfrac{\mathrm{d}}{\mathrm{d}x}(e^{6x}\ln 6x)$ (f) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta\cos 3\theta)$ (g) $\dfrac{\mathrm{d}}{\mathrm{d}t}(\ln t\,\ln 2t)$

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5. The quotient rule for differentiation

The rule for differentiating the quotient of two functions $f(x)$ and $g(x)$ is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

1. Differentiate each of the following with respect to x.

(a) $\dfrac{x}{1-x^2}$ (b) $\dfrac{x^4}{1-x}$ (c) $\dfrac{2-x}{1+2x}$ (d) $\dfrac{3x^2-2x^3}{2x^3+3}$ (e) $\dfrac{1+\sqrt{x}}{\sqrt{x}-x}$

2. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\sin x}{x}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{x^{4/3}}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\dfrac{\theta^2}{\tan 2\theta}\right)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}z}\left(\dfrac{e^z}{\sqrt{z}}\right)$ (e) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2}{\ln 2x}\right)$

3. Find the following derivatives.

(a) $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\sin 2t}{\sin 5t}\right)$ (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{e^{-2x}}{\tan x}\right)$ (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{\cos 3x}\right)$ (d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln 3x}{\ln 4x}\right)$

6. The chain rule for differentiation

The chain rule is used to differentiate a "function of a function":

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(g(x))) = f'(g(x)).g'(x).$$

1. Differentiate each of the following with respect to x.

(a) $(4+3x)^2$ (b) $(1-x^4)^3$ (c) $\frac{1}{(1-2x)^2}$ (d) $\sqrt{1+x^2}$

(e) $\left(x-\frac{1}{x}\right)^{-1/3}$ (f) $(2x^2-3x+5)^{5/2}$ (g) $\sqrt{x-2\sqrt{x}}$ (h) $\frac{1}{\sqrt{4x^2-x^4}}$

2. Find the following derivatives. (Remember the notation for powers of trigonometric functions: "$\sin^2 x$" means $(\sin x)^2$, etc.)

(a) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2\theta)$ (b) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin\theta^2)$ (c) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin(\sin\theta))$ (d) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan(3-4x))$

(e) $\frac{\mathrm{d}}{\mathrm{d}z}(\cos^5 5z)$ (f) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos^3 x}\right)$ (g) $\frac{\mathrm{d}}{\mathrm{d}t}(\sin(2-t-3t^2))$

3. Find the following derivatives. (The notation "$\exp x$" is used rather than "e^x" where it is clearer.)

(a) $\frac{\mathrm{d}}{\mathrm{d}y}\left(\exp(-y^2)\right)$ (b) $\frac{\mathrm{d}}{\mathrm{d}x}(\exp(\cos 3x))$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(e^{3x}))$ (d) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(\sin 4x))$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(\ln 4x))$ (f) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(e^x-e^{-x}))$ (g) $\frac{\mathrm{d}}{\mathrm{d}t}\left(\sqrt{e^{3t}-3\cos 3t}\right)$

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7. Differentiation of functions defined implicitly

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y when x and y are related by the following equations. You will need the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = 1/\frac{\mathrm{d}x}{\mathrm{d}y}$.

(a) $x = y - y^3$ (b) $x = y^2 + \frac{1}{y}$ (c) $x = e^y + e^{2y}$ (d) $x = \ln(y - e^{-y})$

2. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and/or y when x and y are related by the following equations.

(a) $\cos 2x = \tan y$ (b) $x + y^2 = y - x^2$ (c) $y - \sin y = \cos x$

(d) $e^x - x = e^{2y} + 2y$ (e) $x + e^y = \ln x + \ln y$ (f) $y = (x-y)^3$

8. Differentiation of functions defined parametrically

If x and y are both functions of a parameter t, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$$

1. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t when x and y are related by the following pairs of parametric equations.

(a) $x = \sin t, \quad y = \cos t$ (b) $x = t - \frac{1}{t}, \quad y = 1 - t^2$

(c) $x = e^{2t} + t, \quad y = e^t + t^2$ (d) $x = \ln t + t, \quad y = t - \ln t$

2. Find $\frac{\mathrm{d}x}{\mathrm{d}y}$ in terms of t when x and y are related by the following pairs of parametric equations.

(a) $x = 3t + t^3, \quad y = 2t^2 + t^4$ (b) $x = \cos 2t, \quad y = \tan 2t$

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9. Miscellaneous differentiation exercises

1. Find the following derivatives, each of which requires one of the techniques covered in previous sections. You have to decide which technique is required for each derivative!

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 \tan 4x)$ (b) $\frac{\mathrm{d}}{\mathrm{d}t}(\tan^3 4t)$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}(\exp(3\tan 4x))$ (d) $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{3\theta}{\tan 4\theta}\right)$

(e) $\frac{\mathrm{d}}{\mathrm{d}x}(\exp(x - e^x))$ (f) $\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y^4 + y^{-4}}{y + y^{-1}}\right)$ (g) $\frac{\mathrm{d}}{\mathrm{d}x}(2^x x^2)$ (h) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\ln x - x}\right)$

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(5^{-3x}\right)$ (j) $\frac{\mathrm{d}}{\mathrm{d}t}(\ln(\ln t))$ (k) $\frac{\mathrm{d}}{\mathrm{d}z}\left(\ln\left(\frac{1-z}{1+z}\right)^2\right)$

2. Find the following derivatives, which require both the product and quotient rules.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x\cos x}{1 - \cos x}\right)$ (b) $\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^z}{z\ln z}\right)$ (c) $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin 3\theta \cos 2\theta}{\tan 4\theta}\right)$

3. Find the following derivatives, which require the chain rule as well as either the product rule or the quotient rule.

(a) $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-t}\ln(e^t + 1))$ (b) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2 3\theta \, \cos^4 3\theta)$ (c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(\frac{1-x^2}{1+x^2}\right)^{3/2}\right)$

(d) $\frac{\mathrm{d}}{\mathrm{d}\theta}(\exp(\theta\cos\theta))$ (e) $\frac{\mathrm{d}}{\mathrm{d}x}\left((x\ln x)^3\right)$ (f) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\exp\left(\frac{1-x}{1+x}\right)\right)$

(g) $\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{y^2\sqrt{y^2 - 1}}\right)$

4. Find the following derivatives, which require use of the chain rule more than once.

(a) $\dfrac{d}{dx}(\sqrt{1-\cos^3 x})$ (b) $\dfrac{d}{dx}\left(\exp\left((x-x^2)^{1/4}\right)\right)$ (c) $\dfrac{d}{d\theta}\left(\ln\left(\tan\dfrac{1}{\theta}\right)\right)$

5. Find the following second derivatives.

(a) $\dfrac{d^2}{dx^2}(\sqrt{1+x^2})$ (b) $\dfrac{d^2}{dz^2}(\exp(z^2))$ (c) $\dfrac{d^2}{d\theta^2}(\sin^3\theta)$ (d) $\dfrac{d^2}{dx^2}\left(\dfrac{1}{(1-x^4)^4}\right)$

6. Remembering that $\text{cosec}\, x = \dfrac{1}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, find the following derivatives.

(a) $\dfrac{d}{dx}(\text{cosec}\, 2x)$ (b) $\dfrac{d}{d\theta}(\sec^2\theta)$ (c) $\dfrac{d}{dz}(\sqrt{1+\cot z})$

(d) $\dfrac{d}{d\theta}(\text{cosec}^2\theta\cot^3\theta)$ (e) $\dfrac{d}{dx}(\ln(\sec x+\tan x))$ (f) $\dfrac{d}{d\theta}(\tan(\sec\theta))$