3 Session Three Functions, Graphs, Differentiation

3.1 What is a Function?

Functions are an important tool for mathematicians and physicists alike.
See also MathCentre leaflet: • 3.1 What is a function?

A function is a rule that maps each input or set of inputs onto a unique value.

One variable: $f(x)$, $g(t)$: e.g. x^2 , $\sin x$, $\exp(-t^2)$, $t^3 + 3t - 4$.

Two variables: $h(x, y)$, $z(u, v)$: e.g. $x^2 + y^2$, $\sin u \cos v$, e^{xy} .

Multiple variables: $f(x_1, x_2, x_3, \ldots, x_n)$: e.g. $a_1 x_1^{k_1} + \ldots + a_n x_n^{k_n}$.

Functions are often explicitly named so that they can be referred to later.

$y = f(x)$ where $f(x) = x^2$ ... or simply $y = x^2$.

$F = m\,a$, $x = a\sin(\omega t + \phi)$, $v = u + at$, $s = ut + \frac{1}{2}at^2$.

Questions:

1. Given $f(x) = x^2 + 2x + 3$, calculate $f(5)$.
2. Find the range of values for $g(x)$ if $g(x) = \dfrac{1}{x+2}$ and $3 \leq x \leq 8$.
3. Given $h(x) = x^2 + 1$, find the range of values for x such that $5 \leq h(x) \leq 10$.

3.2 Trigonometric Functions

Basic trigonometric functions: $\sin x$, $\cos x$, $\tan x = \dfrac{\sin x}{\cos x}$.

Reciprocals of trigonometric functions:

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x} \; , \quad \sec x = \frac{1}{\cos x} \; , \quad \mathrm{cosec}\, x = \frac{1}{\sin x} \; .$$

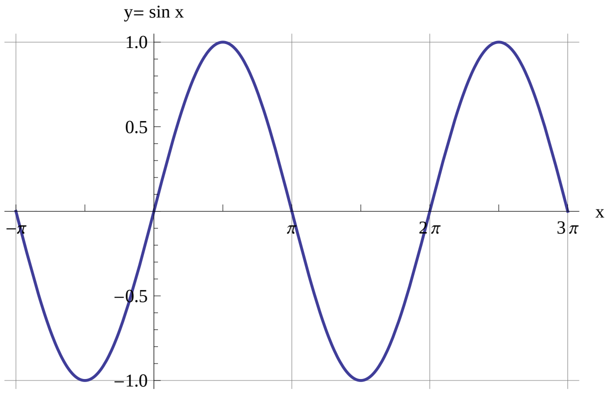
Functions of trigonometric functions: e.g. $\sin^2 x$, $\sin x \cos x$, $a + [\,b\sin(cx + d)\,]$.

Principal values of basic trigonometric functions:

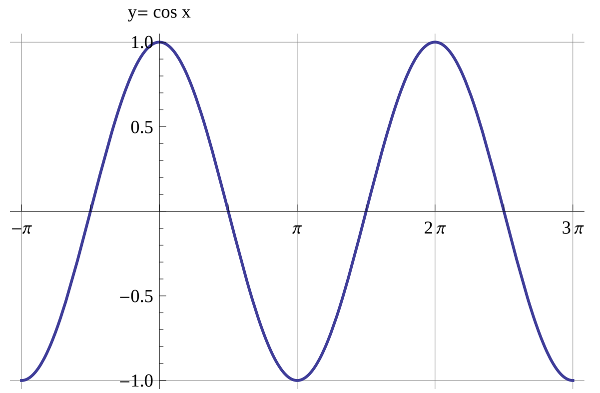
x	$0°$ 0	$30°$ $\pi/6$	$45°$ $\pi/4$	$60°$ $\pi/3$	$90°$ $\pi/2$
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

$$\begin{aligned} \sqrt{2} &= 1.412\ldots \\ \tfrac{1}{\sqrt{2}} &= 0.707\ldots \\ \sqrt{3} &= 1.732\ldots \\ \tfrac{1}{\sqrt{3}} &= 0.577\ldots \\ \tfrac{\sqrt{3}}{2} &= 0.866\ldots \end{aligned}$$

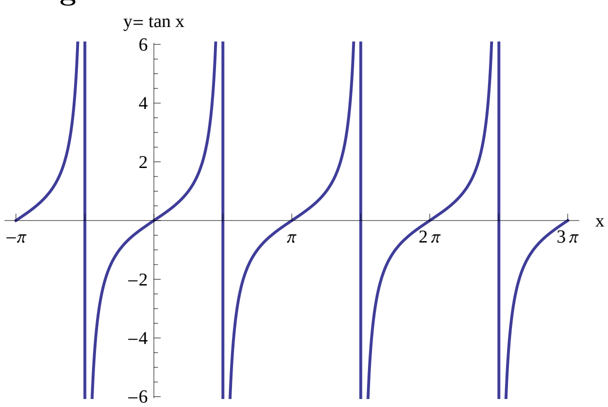
Sine Function



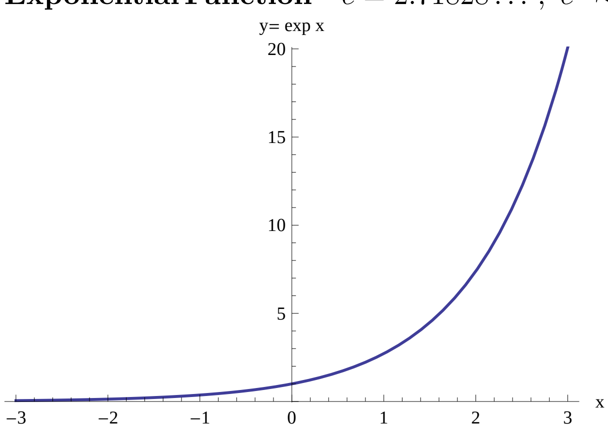
Cosine Function



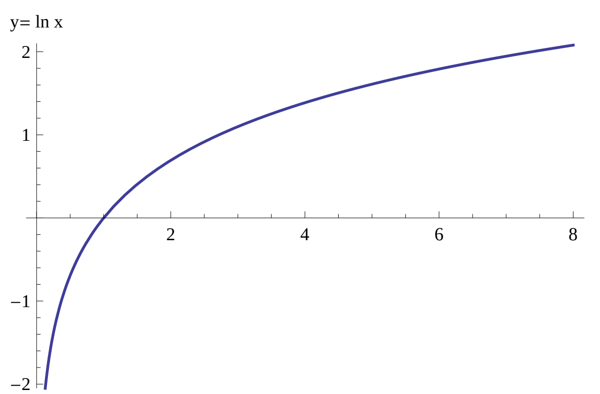
Tangent Function



Exponential Function $e = 2.71828\ldots, \; e^3 \approx 20$



Natural Logarithm Function $\ln 8 \approx 2$

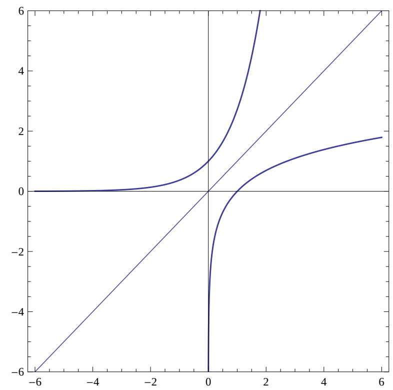


See also MathCentre leaflets: • 3.4 Exponential Function, • 3.7 Logarithm Function

$e =$ 2.7 1828 1828 459045 . . .
$\pi =$ 3.141592 6535 8979 3238 4626 . . .

3.3 Functions and Inverse Functions

Mapping $f(x)$ is functional iff the graph of $y = f(x)$ has at most one y value for any given x value.
Given the function $f(x)$, the inverse $f^{-1}(x)$ is also functional iff the graph of $y = f(x)$ has at most one x value for any given y value.



The graph $y = f^{-1}(x)$ can be obtained from the graph $y = f(x)$ by reflecting it in the line $y = x$.

3.4 Sketching Functions

Example One – Linear Function

It is often useful to sketch the form of a function. Sometimes the form is relatively simple, such as that in the linear relationship that is often used in plotting data for an experiment of form

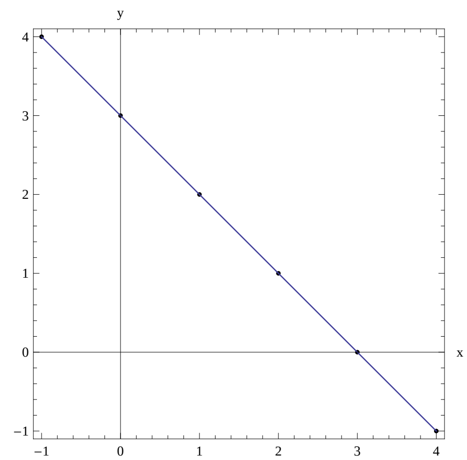
$$y = f(x) = m\,x + c \;\; .$$

This tells us that the gradient $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is of value m, and the intercept on the y-axis is c. Note that it is conventional to plot graphs with the **independent variable** (the input x to the function $f(x)$, the thing we change in the experiment) on the horizontal axis, and the **dependent variable** (the function $f(x)$, the thing we measure in the experiment) on the vertical axis.

You have come across this function so often, that we hope it will be easy to plot. Draw a pair of axes with x on the horizontal axis, and y on the vertical axis. Choose a value of, say, 3 for the intercept c. You now know what value y will be when $x = 0$. Let us take as an example m of -1. You know that the line is a straight line with gradient -1, i.e. the value of y decreases as x increases. If you wish you can plot a table of values of x and the values that y must take for each of these values.

$y = -\,x + 3$

x	0	1	2	3	4	-1
y	3	2	1	0	-1	4



You may wish to calculate the value of x at which the line crosses the x axis. For this condition $y = 0$, and so $-x + 3 = 0$, i.e. $x = 3$. If we want to go the whole hog, we can differentiate y with respect to x and find that the gradient is the constant $m = -1$.
The graph above shows this line, with constant gradient.

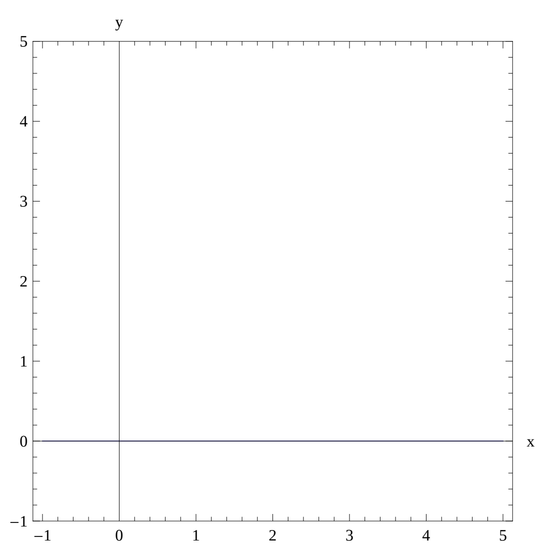
Example Two – Exponential Function

The last example is very well known. Let us now look at one that is a little more involved,

$$y = f(x) = \frac{3}{2}\, e^{-x}$$

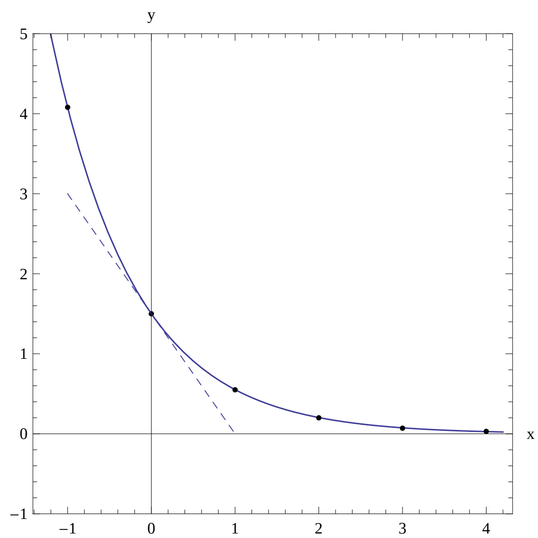
• What value does this function have at $x = 0$?
• What values does it have at other values of x nearby?
• What value does it have as x tends to infinity?
• What value does it tend towards as x becomes very large and negative?
• How does the gradient of the line (the derivative of the function) vary with x? Is it ever zero?
Having answered these questions, try sketching the graph before turning the page.

x	0	1	2	3	4	-1
y						



x	0	1	2	3	4	-1
y	1.5	0.55	0.20	0.07	0.03	4.08

Note, as covered in the first workshop, that e^0 is one. Note that as x becomes large positive, $y = f(x)$ rapidly becomes very small. As x becomes large negative, $y = f(x)$ rapidly becomes very large.



How does this tie in with the gradient?

$$y = f(x) = \frac{3}{2}\,e^{-x}\ , \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3}{2}\,e^{-x}$$

so at $x = 0$ we expect to see a gradient $-3/2$, and the gradient should get less as x becomes larger. The gradient tends to zero as x tends to plus infinity. We do not expect a maximum or minimum in the function between plus and minus infinity. This all ties in with what we see in the graph above, where we have also shown the tangent to the curve at $x = 0$, i.e. the gradient at $x = 0$. This does indeed seem to be -1.5.

Example Three – Sine Function

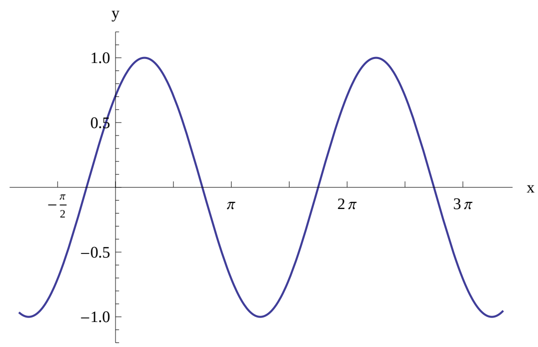
We looked at the sine function in workshop one. We know that when the argument of the sine function is zero, so is its value. Likewise, when the argument of the function is π, 2π, 3π, etc, the value of the function is zero. We know that the maximum of the function is one, and this occurs when the argument of the function is $\pi/2$, $5\pi/2$, etc. You probably remember that the $\sin x$ increases between $x = 0$ and $x = \pi/2$, but we can verify this by looking at the gradient.

$$y = \sin x\ , \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x\ .$$

Admittedly we have only moved to our knowledge of the cosine function to see the gradient of the sine, but please do look carefully at the link between these two functions through differentiation.

Let us try to plot $y = \sin(x + \pi/4)$.

The $\pi/4$ is a phase shift, it will result in the sine curve appearing to have shifted along the axis a little compared to $y = \sin x$. Now, the curve will go through zero when $x + \pi/4 = 0$, i.e. when $x = -\pi/4$. Note that the function will go through zero again each time x increases by π. When $x = 0$, the argument of the function will be $\pi/4$, and $\sin(\pi/4) = 0.707$.



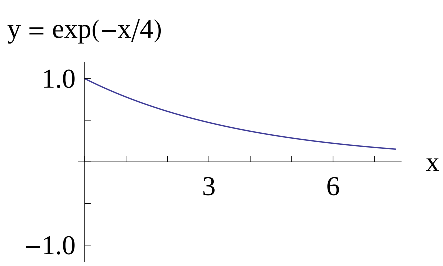
This gives us the graph above. Note the maximum and minimum values of the function, the 2π phase shift between repeats, and the way the curve has been shifted along the x axis compare to $\sin x$. These ideas are very important for your Oscillations and Waves courses.

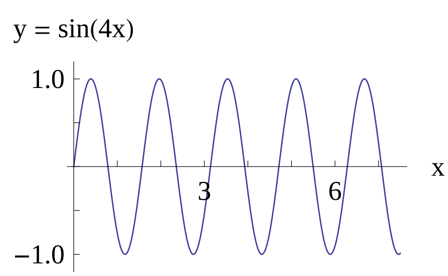
Example Four – One function multiplied by another

If we had tried to plot $4\sin(x + \pi/4)$ above, we can pretty soon work out that the extreme values y would have been $+/-4$. The "amplitude" of the oscillation would have been four units rather than one. But what about something more complicated, like

$$e^{-x/4}\sin(4\,x)\ , \quad x \geq 0\ ?$$

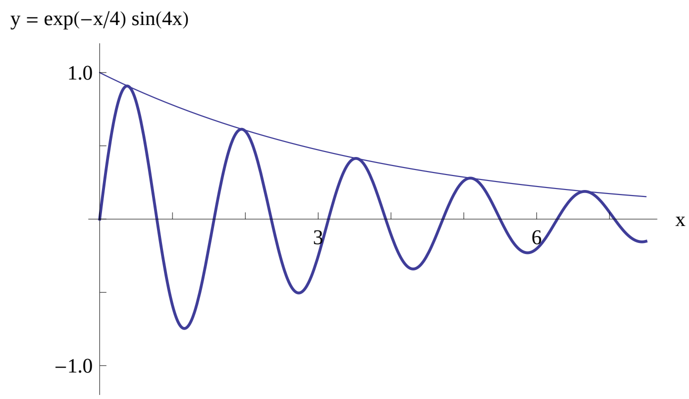
Here we have a sine function multiplied by an exponential decay function.
Using the techniques outlined above, we can first sketch the two functions on their own:





Our original expression had one function multiplied by the other. At each x, our final value should be the value of the exponential function multiplied by that of the sine function. That is indeed how I would deal with the problem using Excel (or Mathematica), but what about if I am sketching the form? As long as the exponential decay is slow compared with the variation of the sine function, then we can think of the sine function as having an amplitude that is unity multiplied by the value of the exponential function at that x.

This is what we show below. The thin line is the exponential function on its own, and it is this that might be thought of as the time-varying amplitude of the sine wave. The solid line is $e^{-x/4}\sin(4x)$. This particular product of functions will be seen to be very important in your Oscillations and Waves courses again.



See also MathCentre leaflet: • 3.2 Graph of a Function

3.5 Basic differentiation

Everyone in the class should be familiar with the basic ideas of differentiation. For a function

$$\begin{aligned} y &= f(x) &= x^n + a\,x^2 + b\,x + c \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= f'(x) &= n\,x^{(n-1)} + 2a\,x + b \end{aligned}$$

MathCentre has revision leaflets on these ideas:

- 8.1 Introduction to differentiation
- 8.2 Table of derivatives
- 8.3 Linearity rules for differentiation

You should also be aware that there is a maximum or minimum where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f'(x) = 0$ and a point of inflexion where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = f''(x) = 0$.

Constant rule, power rule, constant multiple rule, sum or difference rule, derivative of e^x and $\ln x$:

Function	Derivative
C	0
x	1
x^2	$2x$
x^3	$3x^2$
$\vdots$	$\vdots$
x^n	$n\,x^{n-1}$

Function	Derivative
$K\,f(x)$	$K\,f'(x)$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)-g(x)$	$f'(x)-g'(x)$
$e^{k\,x}$	$k\,e^{k\,x}$
$\ln x$	$\dfrac{1}{x}$

Trigonometric functions:

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x} = \sec^2 x$

Function	Derivative
$\operatorname{cosec} x$	$-\operatorname{cosec} x\,\cot x$
$\sec x$	$\sec x\,\tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

Questions: Find the derivatives of the following:

1. $y = 5x^2$
2. $f(x) = 2x^3 + 4x^{-2}$
3. $z = 7y^3 + 6y^2 + 5y + 4$
4. $p(x) = \pi$
5. $g(t) = t^{0.5} - t^{-0.5}$
6. $s = \dfrac{4}{x^3} - \dfrac{2}{3}\sqrt{x}$
7. $y = 2\sin x + 5e^x$
8. $z = 4\tan t - \dfrac{2}{5t^3}$
9. $f(t) = a\,\cos t + b\,\sin t$
10. $g(x) = 4\,\ln x + \ln e^{3x^4}$
11. $h(x) = \sqrt[5]{x^3} - e^{6x}$
12. $y = \ln 4t - \ln \dfrac{t}{3}$