1 Session One - Revision of Basics

1.1 Scientific Notation and Exponents

Note the scientific notation examples below. Note especially where 10^0 fits into the logic:

0.0001	0.003	0.01	0.1	1	10	5100
10^{-4}	3×10^{-3}	10^{-2}	10^{-1}	10^0	10^1	5.1×10^3

If you are happy with what the scientific notation means (eg 10^4 has four tens multiplied together) then in will come as no surprise that

$$10^n \times 10^m = 10^{n+m} \quad \text{and} \quad 10^n/10^m = 10^{n-m} \quad \text{and, more generally,} \quad y^n \times y^m = y^{n+m} \ .$$

Note that in these expressions n and m can be positive or negative or zero. Note also that these exponents are numbers without units or dimensions.

Carrying on, note that

$$x^0 = 1 \ , \quad x^{1/n} = \sqrt[n]{x} \ , \quad (x^n)^m = x^{nm} \ .$$

(Note: More details at www.mathcentre.ac.uk/topics/algebra/powers/ → Quick reference)

1.2 Logarithms

Let us continue with the same ideas to remind ourselves about logarithms. Suppose that a quantity x is expressed as a power of a quantity a, i.e. $x = a^y$. Here a is called the "base" number. The logarithm of x with respect to base a is equal to the exponent to which the base must be raised in order to satisfy the equation $x = a^y$. Thus in this case $\log_a(x) = y$. Note that the subscript after the "log" states the base used. As a is a number without units, then $x = a^y$ is also a number without units. This means that the x in $\log_a(x)$ can not have units.
(More details at www.mathcentre.ac.uk/topics/algebra/logarithms → Quick reference)

The two important bases that we will be using are the base 10, which is called the common logarithm base, and the base $e = 2.171828$ which is known as the natural logarithm base. The latter is so widely used that it is usually not written as $\log_e(x)$ but as $\ln(x)$. In the rest of this document (as on your calculator) we will use $\log(x)$ for log to the base 10 and $\ln(x)$ for log to the base e.

$$\log(1000) = \log(10^3) = 3 \qquad \log(0.001) = \log(10^{-3}) = -3$$

$\log(1378)$ will be a bit more than $3 = \log(10^3)$, but a good deal less than $4 = \log(10^4)$. Your calculator can tell you that it is 3.139...

$$\ln(2.71828) = \ln(e^1) = 1 \quad , \qquad e = 2.71828\,1828\,459045\ldots$$

$\ln(5)$ will be more than 1, but as 5 is less than $e^2 = 7.389$, $\ln(5)$ will be less than 2.
Your calculator will tell you that it is approximately 1.61.

Negative numbers, and zero, cannot be expressed as 10^x, and so it is not possible to take the logarithm of zero or negative numbers.

Properties of logarithms

$\log(x\,y) = \log(x) + \log(y)$	$\ln(x\,y) = \ln(x) + \ln(y)$	(from indices idea above)
$\log(x/y) = \log(x) - \log(y)$	$\ln(x/y) = \ln(x) - \ln(y)$	(likewise)
$\log(x^t) = t\log(x)$	$\ln(x^t) = t\ln(x)$	
$\log(10^a) = a$	$\ln(e^a) = a$	(by definition)
$\log(10) = 1$	$\ln(e) = 1$	(particular example)
$\log(1) = 0$	$\ln(1) = 0$	(particular example)

More examples of logarithms

a) Using the above rules, determine the single logarithm that would represent $\log(6) + \log(2)$.

$$\log(6) + \log(2) \quad = \quad \log(6 \times 2) \quad = \quad \log(12)$$

b) Given that $\log(2) = 0.301$, what is $\log(20)$ without using a calculator?

$$\log(20) \quad = \quad \log(2 \times 10) \quad = \quad \log(2) + \log(10) \quad = \quad \log(2) + 1 \quad = \quad 1.301$$

c) Given that $y = e^{3x}$, find $\ln(y)$.

$$\ln(y) \quad = \quad \ln(e^{3x}) \quad = \quad 3x$$

1.3 Prefixes

We don't always wish to be talking about 3×10^{-3} m, but instead use prefixes. You knew about millimetres in primary school, but now you need to know more prefixes, at least from nano to giga. Note that the centi prefix does not fit into the neat scheme of multiples of one thousand.

multiplier	10^{-15}	10^{-12}	10^{-9}	10^{-6}	10^{-3}	10^{0}	10^{3}	10^{6}	10^{9}	10^{12}
prefix	femto	pico	nano	micro	milli	units	kilo	mega	giga	tera
abbreviation	f	p	n	μ	m		k	M	G	T

Example: How much energy is in a 20 ps long laser pulse with a power of 8 MW?

Energy = power $\times$ time = 20 ps x 8 MW = $20{\times}10^{-12}$ s$\times 8 \times 10^{6}$ W = $160{\times}10^{-6}$ Ws = 160μJ

1.4 Significant Figures

How many figures should you quote in the answer to a calculation or measurement? If we ask you to find ln(17) do we want the 2.833213344 that comes up on your calculator screen? Even that is not the end of it of course.

In a purely mathematical example like this it is difficult to tell, but the use of all ten figures is unlikely to be appropriate. We might guess that the original '17' means a number more than 16.5 and less than 17.5, each of which would give a natural logarithm between 2.803... and 2.862... Thus quoting the final answer to three significant figures (2.83) sounds more than enough well defined. But what if the '17' means 17.00000000000? I suppose the questioner should have told us that, for unless we have reason to know otherwise, the guess above seems sensible.

When we are dealing with real experimental data, it is perhaps easier. If you measure the blade of a wind turbine to be 283 cm long, and you reckon you can measure it to the nearest centimetre, then you should write it as 2.83 m to show this, and not 2.830 which would suggest that you know it to the nearest millimetre. If you want to be more definite about it, you can write $0.283\,\mathrm{m} \pm 0.001\,\mathrm{m}$.

If you then wish to work out the angular speed of the blades tip from the time that it takes to do one revolution ($1\,\mathrm{s} \pm 0.1\,\mathrm{s}$), where do we go?
Well $\omega = 2\pi/T = 2 \times 3.1415927/1 = 6.28318...$ radians per second. How many figures should we quote now? The 10% uncertainty in the period will come through to a 10% uncertainty in the angular velocity, and so we should not wish to quote a number of figures that suggests a much better known measurement than this. In this case this means that we should not use more than two figures in the answer, and so we quote 6.3 radians per second. Better still, we could write 6.3 radians per second $\pm$ 10%. In simple multiplications like this (or divisions), the number of significant figure in the final answer should be about the same as the number of significant figures in the original. If you have two uncertain quantities in a product or quotient, a rough rule of thumb is that the number of significant figures in the answer will be roughly the same as the number of significant figures in the less well known input quantity. You'll find out more about uncertainties and how to work with them from the laboratory course.

1.5 Rearranging and Solving Equations

Simple re-arrangement

If you need to find an "unknown" x in an equation such as $3\,p = b + (x-1)\,c$, you can do straight-forward manipulations to this equation to find x in terms of the other quantities. As long as you do the same thing to both sides of the equation in each step, then the equality will continue to hold.

Example: $3\,p = b + (x-1)\,c$ where we want x as the subject. First, subtract b

$3\,p - b = (x-1)\,c$ from each side, then divide each side by c.

$(3\,p - b)/c = x - 1$ Now add 1 to each side to get:

$x = (3\,p - b)/c + 1$

Quadratic Equations

Sometimes it is just not possible to do simple mathematical manipulations such as those above to get an expression for the desired quantity. An important example is a quadratic equation such as

$$ax^2 + bx + c = 0 \quad .$$

No amount of manipulation of this equation will give us x as the subject. However, it can be shown that the two solutions of this equation for x are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

This (not the method of determining the relation) is something that you are expected to memorise, please. Note that there are two answers, depending on whether the plus or minus sign is taken when evaluating the square root. Both are valid mathematically. In many physical problems one of the two is what you are looking for, and the other is not. You will then need to use your physics knowledge to work out which is which.

Example:

$5y + y^2 = -4$ Find y. Start by rearranging to get it into the standard format,

$y^2 + 5y + 4 = 0$ then use the standard solution format above to get

$$y = \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 4}}{2 \times 1} \quad = \quad \frac{-5 \pm \sqrt{9}}{2}$$

which gives the two solutions for y as -1 (positive root) and -4 (negative root).

Simultaneous Equations

If a problem has n unknowns, then n simultaneous equations are needed to determine those unknowns. For example, we may have two unknowns x and y, with two independent equations relating the two. You may care to think of each equation as describing a line on an xy graph, and the intersection(s) of these two lines will show the place(s) with the solutions for x and y. One way to solve simultaneous equations is to use the first equation to express x in terms of y, and then substitute this expression in for any x appearing in the second equation. This should then give a single equation that involves only the single unknown, y.

Example: Solve the following pair of simultaneous equations:

$$y - x = 2 \quad \text{and} \quad 5x + 9y = 60$$

The first equation tells us that $x = y - 2$, and putting this in the second equations gives

$5(y - 2) + 9y = 60$

ie $5y - 10 + 9y = 60$

so $14y = 70$

ie $y = 5$

Knowing that $y = 5$, we can put this back into $x = y - 2$ to find that $x = 3$. Having now got what you think are the appropriate values for x and y it is well worth putting them back in the two equations to check that you have got your sums right!

Compound Fractions

It is worth pausing to consider that $\frac{a}{b} = a\,\frac{1}{b} = a\,b^{-1}$.

This means for example that $\frac{a/b}{c/d} = \frac{a}{b} \cdot \frac{d}{c}$.

1.6 Handy Formulae

$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$ comes from the usual algebraic rules. Combining the two variants gives:

$a^2 - b^2 = (a+b)(a-b)$ which is something worth remembering.

You are expected to know the following geometrical relations for circles and spheres of radius r:

Circumference of a circle	$2\pi r$	Area of a circle	πr^2
Surface area of a sphere	$4\pi r^2$	Volume of a sphere	$\frac{4}{3}\pi r^3$

The equation for a circle in the (x, y)-plane with centre at (a, b) and with radius r is

$$(x-a)^2 + (y-b)^2 = r^2 \ .$$

$\pi = 3.141592\ 6535\ 8979\ 3238\ 4626\ldots$

1.7 Dimensional Analysis

You are familiar with the units of quantities, such as seconds or metres. You know straight away that three seconds cannot equal four metres. This idea can prove useful. However, we also know that 2.54 cm is (to the precision shown) equal to one inch. In this case the units are not the same, but the "dimensions" are. We will often work with the basic dimensions of length L, time T, mass M, temperature θ, and current I. If two quantities are equal, they MUST have the same dimensions. If two quantities can be added or subtracted, they must have the same dimensions. Stop and think, does it make sense to add three seconds to four metres? No. At the very least, this can be a good way to check your calculations.

Example: One of your friends tries to remember the surface area of a sphere and thinks it may be

$$\frac{4}{3}\pi r^3 \ .$$

You know that the dimensions of area are length times length, i.e. L^2. Four thirds and pi are both plain numbers with no units. The units of the suggested expression are therefore the same as the units of r^3. This will be L^3. These are not the same units as area, and so the suggested expression must be incorrect.
The expression remembered is in fact the volume of a sphere. Note that we can use dimensional analysis to show that the dimension of the expression is correct for a volume, but that dimensional analysis cannot tell us whether or not the four-thirds pi is correct.

Example: It is suggested that for a body travelling in a straight line, the displacement is the "area" underneath the graph of velocity against time. (Note that I have put "area" in inverted commas, as this is an easily misunderstood item, this "area" has the units of the x-axis quantity (time) multiplied by the units of the y-axis quantity (velocity).) For the case of constant acceleration a from initial velocity u, this distance is

$$s = ut + \frac{1}{2}at^2$$

Is this dimensionally correct? The units of displacement s are L.
The units of ut are those of velocity multiplied by time. Velocity we measure as displacement per unit time, i.e. LT^{-1}. Thus the dimensions of ut are $(LT^{-1}) \cdot (T) = L$.
The units of at^2 are $(LT^{-2}) \cdot (T^2) = L$.
This then looks fine dimensionally as on the right hand side we are adding up two quantities of the same dimension, and this dimension is the same as that of the quantity on the left hand side.

Example: What are the dimensions of Force?

We know that we normally measure force in newtons. But what are the dimensions of this quantity? The way I go about determining this is through two quantities being able to be equal only if they have the same dimensions. Force is equal to mass times acceleration. Thus force has the dimensions the same as those of mass multiplied by those of acceleration.

$$F = ma$$

Dimensions of m are M, dimensions of a are LT^{-2}, so dimensions of force are MLT^{-2}.

Example: You are told that the number of remaining nucleii of a radioactive element in a sample is

$$n = N_0\, e^{-kt}$$

where k is the decay constant of 1300 per hour. If you have any doubt about what dimensions or units you should use for t, consider that the exponent must end up being a simple number with no units (see section 1.1). Thus kt must end up as a pure number. If k has dimensions of T^{-1}, then t must have dimensions of T. In this case we can go further. If k has units of "per hour", then we need to have t with units of "hours".

1.8 Trigonometry

Please see the following sheets taken from the MathCentre, with permission.

- 4.1 Degrees and radians
- 4.2 Trigonometrical ratios
- 4.3 Graphs of the trigonometric functions
- 4.4 Trigonometrical identities
- 4.5 Pythagoras' Theorem
- 4.6 The sine rule and cosine rule

Workshop Questions

You should not use a calculator for these questions except where explicitly stated, please.

1. Simplify or evaluate the following:
   (a) $g^2 \cdot g^3$ (b) $x^5 \cdot x^{-6}$ (c) $(y^8)^{0.5}$ (d) $(2^6 \times 5^7)/(50 \times 10^3)$

2. Evaluate e^0

3. Between what two integers must $\log(3163)$ lie?

4. Determine the following expressions as a single logarithm:
   (a) $\log(3) + \log(8)$ (b) $\log(8) + \log(2)$

5. Knowing that $\log(2)$ is 0.301, determine the following without the aid of a calculator:
   (a) $\log(200)$ (b) $\log(4)$

6. Solve for x in $2\log(x) - \log(10x) = 0$

7. Given that $y = e^{5.3}$, find $\ln(y)$.

8. Given that $C = Q/V$, and that Q is 6μC and V is $100\,$mV, what is C?

9. Given that $E = \frac{1}{2}kx^2$, and that k is $3.0\,\mathrm{Nmm^{-1}}$ and x is $4\,$mm, what is E in basic SI units?

10. A cube has sides each $4\,$mm long. Determine the volume of the cube
    (a) by working in millimetres, then converting the determined volume to m^3.
    (b) by converting the length to metres, then determine the volume in m^3.

11. Find x given that $2/x + 7/3 = 5$

12. Rewrite $E = \frac{1}{2}kx^2$ so that x is the subject of the equation.

13. Rewrite $R = (d^2 + L^2)/2d$ so that L is the subject of the equation.

14. Determine x in the following equations:
$x^2 + 3x = 10$ $\quad$ $2x^2 - 5x - 3 = 0$ $\quad$ $2x^4 + x^2 - 10 = 0$

15. Solve for x in the following pairs of simultaneous equations:

$$x + y = 7$$
$$x - y = -1$$

$$96 - 2t = 12x$$
$$t - 15 = 5x$$

16. A spherical ball has a radius of 4 cm. What is its surface area and its volume?

17. A circle has a radius of 4 cm. What is its area and circumference?

18. A right-angled triangle has a hypotenuse of length 5 cm, and one other side of length 3 cm.
(a) What is the length of the third side?
(b) What is the cosine of the angle θ between the hypotenuse and the side of length 3 cm (no calculator needed)?
(c) Again without a calculator, what is $\sin\theta$?
(d) What is $\sin^2\theta + \cos^2\theta$ in this case? (Note: $\sin^2\theta$ means $[\sin\theta]^2$.)

19. A triangle has two sides of length 5 and 4 units, and an angle of 100 degrees between them. Use the cosine rule to determine the length of the third side (calculator allowed).

20. A triangle has its largest internal angle as 140 degrees. The side opposite that angle has length 6 units. One of the other internal angles is 15 degrees. Use the sine rule (calculators allowed) to find the length of the side opposite the 15 degree angle.

21. Without using your calculator, express 30 degrees in radians, including π in the answer.

22. Using your calculator, determine the sine of
(a) 10 degrees (b) 1.57 radians (c) 0.02451 radians (d) 0.12700 radians

Comment on the results of part (c) and (d).

23. Sketch $\sin\theta$ for θ ranging from 0 to 2π radians. Sketch also $\sin^2\theta$ over the same range.

24. Determine the dimensions of pressure.

25. Determine the dimensions of Kinetic Energy and Potential Energy.

26. Is it possible to add 5 N to 6 kg ms^{-2}? Is it possible to multiply 5 N by 6 m?